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Pure discrete spectrum and regular model sets on some non-unimodular substitution tilings

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Substitution tilings with pure discrete spectrum are characterized as regular model sets whose cut-and-project scheme has an internal space that is a product of a Euclidean space and a profinite group. Assumptions made here are that the expansion map of the substitution is diagonalizable and its eigenvalues are all algebraically conjugate with the same multiplicity. A difference from the result of Lee *et al.* [*Acta Cryst.* (2020), A**76**, 600–610] is that unimodularity is no longer assumed in this paper.

1. Introduction

There has been considerable success in studying the structure of tilings with pure discrete spectrum by setting them in the context of model sets (Baake & Moody, 2004; Baake *et al.*, 2007; Strungaru, 2017; Akiyama *et al.*, 2015). However, in general settings, the relation between pure discrete spectrum and model sets is not completely understood and the cut-and-project scheme is usually constructed with an abstract internal space (Baake & Moody, 2004; Strungaru, 2017). Thus it is not easy to understand this relation concretely and get information about the structure from the relation. The notion of inter model sets was introduced by Baake *et al.* (2007) and Lee & Moody (2006) and we know the equivalence between pure discrete spectrum and inter model sets in substitution tilings (Lee, 2007). But there are still some limitations in getting useful information about the cut-and-project scheme (CPS) because the internal space was constructed abstractly. What is the internal space concretely? There was some progress in this direction by Lee *et al.* (2018) and Lee, Akiyama & Lee (2020). However, these papers make various assumptions about substitution tilings such as the expansion map is diagonalizable, the eigenvalues of the expansion map should be algebraically conjugate, the multiplicity of the eigenvalues should be the same, and the expansion map is unimodular. From a long perspective, we aim to gradually eliminate assumptions one by one. As a first step, in this paper we eliminate the assumption of unimodularity.

Our work was inspired by an example of Baake *et al.* (1998), which offers a guide to what the internal space should be. We will look at this in Example 5.10. The present paper is an extension of the result of Lee, Akiyama & Lee (2020) in the sense that the unimodularity condition is removed, and the setting is quite similar.

There are various research works on non-unimodular substitution cases (Baker *et al.*, 2006; Ei *et al.*, 2006; Siegel, 2002) that study symbolic substitution sequences or their geometric substitution tilings in dimension 1. Our definition of non-unimodularity looks slightly different from that defined in those papers. However, if we restrict the substitution tilings to one dimension $\mathbb{R}$, the two definitions are the same.

We have four basic assumptions about a primitive substitution tiling $\mathcal{T}$ on $\mathbb{R}^d$ with an expansion map ϕ:

(i) ϕ is diagonalizable.

(ii) All the eigenvalues of ϕ are algebraically conjugate.

(iii) All the eigenvalues of ϕ have the same multiplicity.

(iv) $\mathcal{T}$ is rigid [see (14) for the definition].

We call these assumptions DAMR. This paper relies heavily on the rigid structure of substitution tilings, and the rigidity property is only known under those assumptions (i), (ii), (iii) together with finite local complexity (Theorem 2.9). In Section 2, we review some definitions and known results that are going to be used in this paper. The main result of this paper shows the following:

Theorem 1.1. Let $\mathcal{T}$ be a repetitive primitive substitution tiling on $\mathbb{R}^d$ with a diagonalizable expansion map ϕ whose eigenvalues are algebraic conjugates with the same multiplicity and let $\mathcal{T}$ be rigid. If $\mathcal{T}$ has pure discrete spectrum, then control point set $\mathcal{C}_j$ of each tile type is a regular model set in the CPS with an internal space which is a product of a Euclidean space and a profinite group, where $\mathcal{C} = (\mathcal{C}_j)_{1\le j\le\kappa}$ is a control point set of $\mathcal{T}$ defined in (7) and the CPS is defined in (35).

In Section 3, we give an outline of the proof of this theorem in some simple case of substitution tilings with expansion map ϕ satisfying the DAMR assumptions defined above. In Section 4, we define an appropriate internal space and construct a CPS under the DAMR assumptions. Then we discuss the projected point sets $E_{\delta,k}$ of neighbourhood bases of a topology in the internal space. In Section 6, under the assumption of pure discrete spectrum of $\mathcal{T}$, we look at how the projected point sets $E_{\delta,k}$ and the translation vector set Ξ of the same types of tiles in $\mathcal{T}$ are related [see (8)]. Using the equivalent property 'algebraic coincidence' for pure discrete spectrum, we provide arguments to show that we actually have regular model sets.

2. Definitions and known results

We consider a primitive substitution tiling $\mathcal{T}$ on $\mathbb{R}^d$ with expansion map ϕ satisfying the DAMR assumptions defined above. In this section, we recall some definitions and results that we are going to use in the later sections.

2.1. Tilings

We consider a set of types (or colours) $\{1, \ldots, \kappa\}$, which we fix once and for all. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$, which is the closure of its interior, and $i = l(T) \in \{1, \ldots, \kappa\}$ is the type of T. A *tiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \cup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors.

Given a tiling $\mathcal{T}$, a finite set of tiles of $\mathcal{T}$ is called a $\mathcal{T}$*-patch*. Recall that a tiling $\mathcal{T}$ is said to be *repetitive* if the occurrence of every $\mathcal{T}$-patch is relatively dense in space. We say that a tiling $\mathcal{T}$ has *finite local complexity* (FLC) if for every $R > 0$ there are only finitely many translational classes of $\mathcal{T}$-patches whose support lies in some ball of radius R up to translations.

2.2. Delone κ-sets

A κ*-set* in $\mathbb{R}^d$ is a subset $\mathbf{\Lambda} = \Lambda_1 \times \ldots \times \Lambda_\kappa \subset \mathbb{R}^d \times \ldots \times \mathbb{R}^d$ (κ copies) where $\Lambda_i \subset \mathbb{R}^d$ and κ is the number of colours. We also write $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_\kappa) = (\Lambda_i)_{i\le\kappa}$. Recall that a Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$. We say that $\mathbf{\Lambda} = (\Lambda_i)_{i\le\kappa}$ is a *Delone* κ*-set* in $\mathbb{R}^d$ if each Λ_i is Delone and $\mathrm{supp}(\mathbf{\Lambda}) := \cup_{i=1}^{\kappa}\Lambda_i \subset \mathbb{R}^d$ is Delone. The type (or colour) of a point x in the Delone κ-set Λ is i if $x \in \Lambda_i$ with $1 \le i \le \kappa$.

A Delone set Λ is called a *Meyer set* in $\mathbb{R}^d$ if $\Lambda - \Lambda$ is uniformly discrete, which is equivalent to saying that $\Lambda - \Lambda = \Lambda + F$ for some finite set F (see Meyer, 1972; Lagarias, 1996; Moody, 1997). If Λ is a Delone κ-set and $\mathrm{supp}(\mathbf{\Lambda})$ is a Meyer set, we say that Λ is a Meyer κ-set.

2.3. Substitutions

We say that a linear map $\phi: \mathbb{R}^d \to \mathbb{R}^d$ is *expansive* if there is a constant $c > 1$ with

$$d(\phi\mathbf{x}, \phi\mathbf{y}) \ge c d(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ under some metric d on $\mathbb{R}^d$ compatible with the standard topology.

Definition 2.1. Let $\mathcal{A} = \{T_1, \ldots, T_\kappa\}$ be a finite set of tiles on $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. Denote by $\mathcal{P}_\mathcal{A}$ the set of patches made of tiles each of which is a translate of one of the T_i's. We say that $\omega : \mathcal{A} \to \mathcal{P}_\mathcal{A}$ is a *tile-substitution* (or simply *substitution*) with an expansive map ϕ if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le \kappa$, such that

$$\omega(T_j) = \{\mathbf{u} + T_i : \ \mathbf{u} \in \mathcal{D}_{ij}, \ i = 1, \ldots, \kappa\} \qquad (1)$$

with

$$\phi A_j = \bigcup_{i=1}^{\kappa}(\mathcal{D}_{ij} + A_i) \quad \text{for each } j \le \kappa. \qquad (2)$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty.

The substitution (1) is extended to all translates of prototiles by $\omega(\mathbf{x} + T_j) = \phi\mathbf{x} + \omega(T_j)$, and to patches and tilings by $\omega(\mathcal{P}) = \cup\{\omega(T) : T \in \mathcal{P}\}$. The substitution ω can be iterated, producing larger and larger patches $\omega^k(\mathcal{P})$. A tiling $\mathcal{T}$ satisfying $\omega(\mathcal{T}) = \mathcal{T}$ is called a *fixed point of the tile-substitution* or a *substitution tiling with expansion map* ϕ. It is known (and easy to see) (Solomyak, 1997) that one can always find a periodic point for ω in the tiling dynamical hull, *i.e.* $\omega^N(\mathcal{T}) = \mathcal{T}$ for some $N \in \mathbb{N}$. In this case we use ω^N in the place of ω to obtain a fixed point tiling. The *substitution* $\kappa \times \kappa$ matrix S of the tile-substitution is defined by $\mathsf{S}(i, j) = \#\mathcal{D}_{ij}$. We say that the substitution tiling $\mathcal{T}$ is *primitive* if there is an $\ell > 0$ for which S^ℓ has no zero entries, where S is the substitution matrix.

When there exists a monic polynomial $P(x)$ over $\mathbb{Z}$ with the minimal degree satisfying $P(\phi) = 0$, we call the polynomial the *minimal polynomial of* ϕ *over* $\mathbb{Z}$. We say that ϕ is *unimodular* if the minimal polynomial of ϕ over $\mathbb{Z}$ has constant term ± 1; that is to say, the product of all roots of the minimal polynomial of ϕ is ± 1. If the constant term in the minimal polynomial of ϕ is not ± 1, then we say that ϕ is *non-unimodular*.

Note that for $M \in \mathbb{N}$,

$$\phi^M A_j = \bigcup_{i=1}^{\kappa} (\mathcal{D}_{ij}^M + A_i) \quad \text{for } j \le \kappa,$$

where

$$(\mathcal{D}^M)_{ij} = \bigcup_{k_1, k_2, \ldots, k_{(M-1)} \le \kappa} (\mathcal{D}_{ik_1} + \phi \mathcal{D}_{k_1 k_2} + \cdots + \phi^{M-1} \mathcal{D}_{k_{(M-1)} j}). \tag{3}$$

Definition 2.2. $\mathbf{\Lambda} = (\Lambda_i)_{i \le \kappa}$ is called a *substitution Delone* κ*-set* if Λ is a Delone κ-set and there exist an expansive map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ and finite sets $\mathcal{D}_{ij}$ for $i, j \le \kappa$ such that

$$\Lambda_i = \bigcup_{j=1}^{\kappa} (\phi \Lambda_j + \mathcal{D}_{ij}), \quad i \le \kappa, \tag{4}$$

where the unions on the right-hand side are disjoint.

Definition 2.3. For a substitution Delone κ-set $\mathbf{\Lambda} = (\Lambda_i)_{i \le \kappa}$ satisfying (4), define a matrix $\Phi = (\Phi_{ij})_{i,j=1}^{\kappa}$ whose entries are finite (possibly empty) families of linear affine transformations on $\mathbb{R}^d$ given by

$$\Phi_{ij} = \{f : \mathbf{x} \mapsto \phi \mathbf{x} + \mathbf{u} \mid \mathbf{u} \in \mathcal{D}_{ij}\}.$$

Define $\Phi_{ij}(\mathcal{X}) := \cup_{f \in \Phi_{ij}} f(\mathcal{X})$ for $\mathcal{X} \subset \mathbb{R}^d$. For a κ-set $(\mathcal{X}_i)_{i \le \kappa}$ let

$$\Phi\big((\mathcal{X}_i)_{i \le \kappa}\big) = \left(\bigcup_{j=1}^{\kappa} \Phi_{ij}(\mathcal{X}_j) \right)_{i \le \kappa}. \tag{5}$$

Thus $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$ by definition. We say that Φ is a κ*-set substitution*. Let

$$S(\Phi) = (\text{card } \Phi_{ij})_{ij}$$

denote the *substitution matrix* corresponding to Φ.

Definition 2.4. (Mauduit, 1989.) An algebraic integer θ is a real Pisot number if it is greater than 1 and all its Galois conjugates are less than 1 in modulus, and a complex Pisot number if every Galois conjugate, except the complex conjugate $\overline{\theta}$, has modulus less than 1. A set of algebraic integers $\Theta = \{\theta_1, \ldots, \theta_r\}$ is a *Pisot family* if for every $1 \le j \le r$, every Galois conjugate η of θ_j, with $|\eta| \ge 1$, is contained in Θ.

For $r = 1$, with θ_1 real and $|\theta_1| > 1$, this reduces to $|\theta_1|$ being a real Pisot number, and for $r = 2$, with θ_1 non-real and $|\theta_1| > 1$, to θ_1 being a complex Pisot number.

2.4. Pure discrete spectrum and algebraic coincidence

Let $X_{\mathcal{T}}$ be the collection of tilings on $\mathbb{R}^d$ each of whose patches is a translate of a $\mathcal{T}$-patch. In the case that $\mathcal{T}$ has FLC, there is a well known metric δ on the tilings: for a small $\epsilon > 0$ two tilings $\mathcal{S}_1, \mathcal{S}_2$ are ϵ-close if $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on the ball of radius ϵ^{-1} around the origin, after a translation of size less than ϵ (see Schlottmann, 2000; Radin & Wolff, 1992; Lee *et al.*, 2003). Then

$$X_{\mathcal{T}} = \overline{\{-\mathbf{h} + \mathcal{T} : \mathbf{h} \in \mathbb{R}^d\}}$$

where the closure is taken in the topology induced by the metric δ.

It is known that a dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ with a primitive substitution tiling $\mathcal{T}$ always has a unique ergodic measure μ in the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ (see Solomyak, 1997; Lee *et al.*, 2003). We consider the associated group of unitary operators $\{T_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^d}$ on $L^2(X_{\mathcal{T}}, \mu)$:

$$T_g(\mathcal{T}') = g(-\mathbf{x} + \mathcal{T}').$$

Every $g \in L^2(X_{\mathcal{T}}, \mu)$ defines a function on $\mathbb{R}^d$ by $\mathbf{x} \mapsto \langle T_{\mathbf{x}} g, g \rangle$. This function is positive definite on $\mathbb{R}^d$, so its Fourier transform is a positive measure σ_g on $\mathbb{R}^d$ called the *spectral measure* corresponding to g. The dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ is said to have *pure discrete spectrum* if σ_g is pure point for every $g \in L^2(X_{\mathcal{T}}, \mu)$. We also say that $\mathcal{T}$ has pure discrete spectrum if the dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ has pure discrete spectrum.

The notion of pure discrete spectrum of the dynamical system is quite closely connected wtih the notion of algebraic coincidence in Definition 2.6. For this we start by introducing control points. There is a standard way to choose distinguished points in the tiles of a primitive substitution tiling so that they form a ϕ-invariant Delone κ-set. They are called *control points* (Thurston, 1989; Praggastis, 1999), which are defined below.

Definition 2.5. Let $\mathcal{T}$ be a primitive substitution tiling with an expansion map ϕ. For every $\mathcal{T}$-tile T, we choose a tile $\gamma(T)$ in the patch $\omega(T)$; for all tiles of the same type in $\mathcal{T}$, we choose $\gamma(T)$ with the same relative position [*i.e.* if $S = \mathbf{x} + T$ for some two tiles $S, T \in \mathcal{T}$ then $\gamma(S) = \mathbf{x} + \gamma(T)$]. This defines a map $\gamma : \mathcal{T} \to \mathcal{T}$ called the *tile map*. Then we define the *control point* for a tile $T \in \mathcal{T}$ by

$$\{c(T)\} = \bigcap_{m=0}^{\infty} \phi^{-m}(\gamma^m(T)). \tag{6}$$

The control points satisfy the following: (*a*) $T' = T + c(T') - c(T)$, for any tiles T, T' of the same type; (*b*) $\phi(c(T)) = c(\gamma(T))$, for $T \in \mathcal{T}$.

Let

$$\mathcal{C} := \mathcal{C}(\mathcal{T}) = (\mathcal{C}_i)_{i \le \kappa} = \{c(T) : T \in \mathcal{T}\} \tag{7}$$

be a set of control points of the tiling $\mathcal{T}$ in $\mathbb{R}^d$. Let us denote $\cup_{i \le \kappa} \mathcal{C}_i$ by supp $\mathcal{C}$.

For tiles of any tiling $\mathcal{S} \in X_{\mathcal{T}}$, the control points have the same relative position as in $\mathcal{T}$-tiles. The choice of control points is non-unique, but there are only finitely many possibilities, determined by the choice of the tile map. Let

$$\Xi = \bigcup_{i=1}^{\kappa} (\mathcal{C}_i - \mathcal{C}_i). \tag{8}$$

Since the substitution tiling $\mathcal{T}$ is primitive, it is possible to assume that the substitution matrix S is positive taking S^k if necessary. So we consider a tile map

$$\gamma : \mathcal{T} \to \mathcal{T} \tag{9}$$

with the property that for every $T \in \mathcal{T}$, the tile $\gamma(T)$ has the same tile type in $\mathcal{T}$. That is to say, for every $T, S \in \mathcal{T}$, $\gamma(T) = x + \gamma(S)$, where $\gamma(T), \gamma(S) \in \mathcal{T}$ and $x \in \Xi$. Then for any $T, S \in \mathcal{T}$,

$$c(\gamma(T)) - c(\gamma(S)) \;\in\; \Xi.$$

In order to have $0 \in \mathcal{C}_j$ for some $j \leq \kappa$ and $\mathcal{C}_j \subset \Xi$, we define the tile map as follows. It is known that there exists a finite generating patch $\mathcal{P}$ for which $\lim_{n\to\infty} \omega^n(\mathcal{P}) = \mathcal{T}$ (Lagarias & Wang, 2003). Although it was defined there for primitive substitution *point sets*, it is easy to see that the same property holds for primitive substitution tilings. We call the finite patch $\mathcal{P}$ the *generating tile set*. When we apply the substitution infinitely many times to the generating tile set $\mathcal{P}$, we obtain the whole substitution tiling. So there exists $n \in \mathbb{N}$ such that the nth iteration of the substitution to the generating tile set covers the origin. We choose a tile R in a patch $\omega^n(\mathcal{P})$ which contains the origin, where $R = \boldsymbol{a} + T_j$ for some $1 \leq j \leq \kappa$. Then there exists a fixed tile $S \in \mathcal{P}$ such that $R \in \omega^n(S)$. Replacing the substitution ω by ω^n, we can define a tile map γ so that

$$\begin{cases} \gamma(T) \text{ is a } j\text{-type tile in } \omega^n(T) & \text{if } T \in \mathcal{T} \text{ with } T \neq S \\ \gamma(S) = R. \end{cases}$$

Then $0 \in \mathcal{C}_j$ by the definition of the control point sets and so $\mathcal{C}_j \subset \Xi$. Since $\phi(c(T)) = c(\gamma(T))$ for any $T \in \mathcal{T}$,

$$\phi(\mathcal{C}_i) \subset \mathcal{C}_j \quad \text{for any } i \leq \kappa.$$

This implies that

$$\phi\left\langle \bigcup_{i\leq\kappa} \mathcal{C}_i \right\rangle_{\mathbb{Z}} \;\subset\; \langle \Xi \rangle_{\mathbb{Z}}. \tag{10}$$

Definition 2.6. (Lee, 2007.) Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with an expansive map ϕ and let $\mathcal{C} = (\mathcal{C}_i)_{i\leq\kappa}$ be a corresponding control point set. We say that $\mathcal{C}$ admits an *algebraic coincidence* if there exists $M \in \mathbb{Z}_+$ and $\xi \in \mathcal{C}_i$ for some $1 \leq i \leq \kappa$ such that

$$\xi + \phi^M \Xi \;\subset\; \mathcal{C}_i.$$

Note that if the algebraic coincidence is assumed, then for some $k \in \mathbb{Z}_+$,

$$\phi^k \Xi - \phi^k \Xi \;\subset\; \mathcal{C}_i - \mathcal{C}_i \;\subset\; \Xi. \tag{11}$$

Theorem 2.7. [Theorem 3.13 (Lee, 2007), Theorem 2.6 (Lee, Akiyama & Lee, 2020).] Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with an expansive map ϕ and $\mathcal{C} = (\mathcal{C}_i)_{i\leq\kappa}$ be a control point set of $\mathcal{T}$. Suppose that all the eigenvalues of ϕ are algebraic integers. Then $\mathcal{T}$ has pure discrete spectrum if and only if $\mathcal{C}$ admits an algebraic coincidence.

2.5. CPS

We use a standard definition for a CPS and model sets (see Baake & Grimm, 2013). For convenience, we give the definition for our setting.

Definition 2.8. A *CPS* consists of a collection of spaces and mappings as follows:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ & & \widetilde{L} & & \end{array} \tag{12}$$

where $\mathbb{R}^d$ is a real Euclidean space, H is a locally compact Abelian group, π_1 and π_2 are the canonical projections, $\widetilde{L} \subset \mathbb{R}^d \times H$ is a lattice, *i.e.* a discrete subgroup for which the quotient group $(\mathbb{R}^d \times H)/\widetilde{L}$ is compact, $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H. For a subset $V \subset H$, we define

$$\curlywedge(V) := \{\pi_1(\mathbf{x}) \in \mathbb{R}^d : \mathbf{x} \in \widetilde{L}, \pi_2(\mathbf{x}) \in V\}.$$

Here the set V is called *a window* of $\curlywedge(V)$. A subset Λ of $\mathbb{R}^d$ is called a *model set* if Λ can be of the form $\curlywedge(W)$, where $W \subset H$ has non-empty interior and compact closure in the setting of the CPS in (12). The model set Λ is *regular* if the boundary of W

$$\partial W = \overline{W} \setminus W^\circ$$

is of (Haar) measure 0. We say that $\boldsymbol{\Lambda} = (\Lambda_i)_{i\leq\kappa}$ is a *model κ-set* (respectively, *regular model κ-set*) if each Λ_i is a model set (respectively, regular model set) with respect to the same CPS.

2.6. Rigid structure on substitution tilings

The structure of a module generated by the control points is known only for the diagonalizable case for ϕ whose eigenvalues are algebraically conjugate with the same multiplicity given by Lee & Solomyak (2012). We need to use the structure of the module in the subsequent sections. Thus we will have the same assumptions.

Let J be the multiplicity of each eigenvalue of ϕ and assume that the number of distinct eigenvalues of ϕ is m. For $1 \leq j \leq J$, we define $\boldsymbol{a}_j \in \mathbb{R}^d$ such that for each $1 \leq k \leq d$,

$$(\boldsymbol{a}_j)_k = \begin{cases} 1 & \text{if } (j-1)m + 1 \leq k \leq jm; \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

We recall the following theorem for the module structure of the control point sets. Although the theorem is not explicitly stated by Lee & Solomyak (2012), it can be read off from their Theorem 4.1 and Lemma 6.1.

Theorem 2.9. (Lee & Solomyak, 2012.) Let $\mathcal{T}$ be a repetitive primitive substitution tiling on $\mathbb{R}^d$ with an expansion map ϕ. Assume that $\mathcal{T}$ has FLC, ϕ is diagonalizable, and all the eigenvalues of ϕ are algebraically conjugate with the same multiplicity J. Then there exists a linear isomorphism $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\sigma\phi = \phi\sigma \quad \text{and} \quad \sigma(\operatorname{supp} \mathcal{C}(\mathcal{T})) \subset \mathbb{Z}[\phi]\boldsymbol{a}_1 + \cdots + \mathbb{Z}[\phi]\boldsymbol{a}_J,$$

where $\boldsymbol{a}_j$, $1 \le j \le J$, are given as (13) and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_J \in \sigma(\operatorname{supp} \mathcal{C}(\mathcal{T}))$.

Note here that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_J$ are linearly independent over $\mathbb{Z}[\phi]$. A tiling $\mathcal{T}$ is said to be *rigid* if $\mathcal{T}$ satisfies the result of Theorem 2.9; that is to say, there exists a linear isomorphism $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\sigma\phi = \phi\sigma \quad \text{and} \quad \sigma(\operatorname{supp} \mathcal{C}(\mathcal{T})) \subset \mathbb{Z}[\phi]\boldsymbol{a}_1 + \cdots + \mathbb{Z}[\phi]\boldsymbol{a}_J, \tag{14}$$

where $\boldsymbol{a}_j$, $1 \le j \le J$, are given as (13).

As an example of a substitution tiling with the rigidity property, let us look at the Frank–Robinson substitution tiling (Frank & Robinson, 2006) (Fig. 1).

Take the tile-substitution

$$\begin{aligned}\phi A_1 = {} & (A_1 + (2,2)) \cup (A_2 + (2,0)) \cup (A_2 + (2,1)) \\ & \cup (A_2 + (0, b+2)) \cup (A_3 + (0,2)) \cup (A_3 + (1,2)) \\ & \cup (A_3 + (b+2, 0)) \cup A_4 \cup (A_4 + (1,0)) \\ & \cup (A_4 + (0,1)) \cup (A_4 + (1,1)) \cup (A_4 + (b+2, b)) \\ & \cup (A_4 + (b+2, b+1)) \cup (A_4 + (b+2, b+2)) \\ & \cup (A_4 + (b+1, b+2)) \cup (A_4 + (b, b+2)),\end{aligned}$$

$$\begin{aligned}\phi A_2 = {} & A_1 \cup (A_3 + (b, 0)) \cup (A_3 + (b+1, 0)) \\ & \cup (A_3 + (b+2, 0)),\end{aligned}$$

$$\begin{aligned}\phi A_3 = {} & A_1 \cup (A_2 + (0, b)) \cup (A_2 + (0, b+1)) \\ & \cup (A_2 + (0, b+2)),\end{aligned}$$

$$\phi A_4 = A_1,$$

where b is the largest root of $x^2 - x - 3 = 0$ and

$$\phi = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

Then it gives a primitive substitution tiling $\mathcal{T}$. Note that b is not a Pisot number. It was shown by Frank & Robinson (2006) that $\mathcal{T}$ does not have FLC. One can observe that each set of translation vectors satisfies $\mathcal{D}_{ij} \subset \mathbb{Z}[\phi](1,0) + \mathbb{Z}[\phi](0,1)$. Thus

$$\Xi(\mathcal{T}) \subset \mathbb{Z}[\phi](1,0) + \mathbb{Z}[\phi](0,1),$$

whence the rigidity holds.

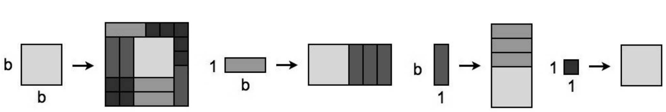


Figure 1
The Frank–Robinson tiling substitution.

3. Outline of the proof of Theorem 1.1

We provide a brief outline of the proof of Theorem 1.1 for the simpler case of repetitive primitive substitution tilings $\mathcal{T}$ on $\mathbb{R}$ with an expansion factor λ ($\phi = \lambda$):

(*a*) λ is non-unimodular,
(*b*) λ is a real Pisot number which is not an integer,
(*c*) $\mathcal{T}$ has FLC,
(*d*) $\mathcal{T}$ has pure discrete spectrum.

Let $P(x)$ be the minimal polynomial of ϕ over $\mathbb{Z}$ for which $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. Let $\lambda, \lambda_2, \ldots, \lambda_n$ be all the roots of the equation $P(x) = 0$, where the absolute values of $\lambda_2, \ldots, \lambda_n$ are all less than 1. Using the rigidity of Theorem 2.9, we get up to an isomorphism

$$\operatorname{supp} \mathcal{C}(\mathcal{T}) \subset \mathbb{Z}[\lambda].$$

Using the algebraic conjugates $\lambda_2, \ldots, \lambda_n$ of λ whose absolute values are less than 1, we consider a Euclidean space $\mathbb{R}^{n-1}$ and the map

$$\Psi_0 : \mathbb{Z}[\lambda] \to \mathbb{R}^{n-1}, \quad P(\lambda) \mapsto P(D)\boldsymbol{\beta},$$

where

$$D = \begin{bmatrix} \lambda_2 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & \lambda_n \end{bmatrix} \quad \text{and } \boldsymbol{\beta} = (1, \ldots, 1) \in \mathbb{R}^{n-1}.$$

For the case of non-unimodular λ, we construct a profinite group below. We remark that if λ is unimodular, then the profinite group is trivial so that Theorem 1.1 can be covered by the work of Lee, Akiyama & Lee (2020). Let $L = \mathbb{Z}[\lambda]$. From the non-unimodularity of λ, $a_0 \ne \pm 1$. So $\lambda L \not\subseteq L$. Note that $\{1, \lambda, \ldots, \lambda^{n-1}\}$ is a basis of L as a free $\mathbb{Z}$-module. Consider the map

$$\begin{aligned} & \pi : L \to \mathbb{Z}^n, \ \pi(\mathbf{v}) = (c_1, \ldots, c_n) \\ & \text{where } \mathbf{v} = c_1 + c_2\lambda + \cdots + c_n\lambda^{n-1}. \end{aligned} \tag{15}$$

This gives an isomorphism of the $\mathbb{Z}$-module between L and $\mathbb{Z}^n$. Let

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

be the companion matrix of $P(x)$. Then

$$\pi(\lambda\mathbf{v}) = M\pi(\mathbf{v}) \quad \text{for any } \mathbf{v} \in L.$$

Notice that M acts on $\mathbb{Z}^n$ and the roots of the minimal polynomial of M over $\mathbb{Z}$ are exactly $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since $\lambda L \not\subseteq L$, $M\mathbb{Z}^n \not\subseteq \mathbb{Z}^n$. Here we consider a profinite group

$$\overleftarrow{\mathbb{Z}^n_M} := \lim_{\leftarrow} \mathbb{Z}^n/M^k\mathbb{Z}^n$$
$$= \mathbb{Z}^n/M\mathbb{Z}^n \leftarrow \mathbb{Z}^n/M^2\mathbb{Z}^n \leftarrow \mathbb{Z}^n/M^3\mathbb{Z}^n \leftarrow \cdots$$
$$= \{(\mathbf{x}_1 + M\mathbb{Z}^n, \mathbf{x}_2 + M^2\mathbb{Z}^n, \mathbf{x}_3 + M^3\mathbb{Z}^n, \ldots) \mid$$
$$\mathbf{x}_1 \in \mathbb{Z}^n, \mathbf{x}_k \in \mathbf{x}_{k-1} + M^{k-1}\mathbb{Z}^n \text{ for each integer } k \geq 2\}. \tag{16}$$

Since $\mathbb{Z}^n$ embeds in $\overleftarrow{\mathbb{Z}^n_M}$, we can identify $\mathbb{Z}^n$ with its image in $\overleftarrow{\mathbb{Z}^n_M}$. Consider the following map:

$$\Psi : L \to \mathbb{R}^{n-1} \times \overleftarrow{\mathbb{Z}^n_M}, \quad \Psi(\mathbf{x}) = (\Psi_0(\mathbf{x}), \pi(\mathbf{x})).$$

Now we construct a CPS whose physical space is $\mathbb{R}$ and internal space is $\mathbb{R}^{n-1} \times \overleftarrow{\mathbb{Z}^n_M}$:

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times (\mathbb{R}^{n-1} \times \overleftarrow{\mathbb{Z}^n_M}) & \xrightarrow{\pi_2} & \mathbb{R}^{n-1} \times \overleftarrow{\mathbb{Z}^n_M} \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & \Psi(L) \\ & & \Downarrow & & \\ \mathbf{x} & \longleftarrow & (\mathbf{x}, \Psi(\mathbf{x})) & \longmapsto & \Psi(\mathbf{x})\,. \end{array} \tag{17}$$

Under the assumption of pure discrete spectrum of $\mathcal{T}$, we know that an algebraic coincidence occurs by Theorem 2.7. So there exist $M \in \mathbb{Z}_+$ and $\xi \in \mathcal{C}_i$ for some $1 \leq i \leq \kappa$ such that

$$\xi + \lambda^M \Xi \subset \mathcal{C}_i,$$

where Ξ is the set of translational vectors which translate a tile to the same type of tile in $\mathcal{T}$ as given in (8). Notice that $B_\delta(0) \times M^k\, \overleftarrow{\mathbb{Z}^n_M}$ is a basis element in the locally compact abelian group $\mathbb{R}^{n-1} \times \overleftarrow{\mathbb{Z}^n_M}$ where $B_\delta(0)$ is a ball of radius δ around 0 in $\mathbb{R}^{n-1}$. We let $E_{\delta,k} = \wedge(B_\delta(0) \times M^k\, \overleftarrow{\mathbb{Z}^n_M})$ be the projected point set in $\mathbb{R}$ coming from a window $B_\delta(0) \times M^k\, \overleftarrow{\mathbb{Z}^n_M}$. It is important to understand the relation between $E_{\delta,k}$ and Ξ. We discuss this in Section 4.2 (see also Lee *et al.*, 2018; Lee, Akiyama & Lee, 2020). From this relation, together with algebraic coincidence, we can view the control point set of $\mathcal{T}$ as a model set. Using Keesling's argument (Keesling, 1999), we show that the control point set of $\mathcal{T}$ is actually a regular model set.

4. Construction of a CPS

We aim to prove that the structure of pure discrete spectrum in a substitution tiling can be described by a regular model set which comes from a CPS with the internal space that is a product of a Euclidean space and a profinite group. From Lee & Solomyak (2019), under the assumption of pure discrete spectrum, the control point set of the substitution tiling has the Meyer property and so has FLC. In general settings which are not substitution tilings, it is hard to expect that pure discrete spectrum implies neither the Meyer property nor FLC (Lee, Lenz *et al.*, 2020).

The setting that we consider here is a primitive substitution tiling $\mathcal{T}$ on $\mathbb{R}^d$ with an expansion map ϕ which satisfies the DAMR assumptions. Changing the tile substitution if necessary, we can assume that ϕ is a diagonal matrix without loss of generality.

Under the assumption of DAMR, it is also known from Lee & Solomyak (2012, 2019) that the control point set of the substitution tiling has the Meyer property if and only if the eigenvalues of ϕ form a Pisot family. In our setting, there is no algebraic conjugate η with $|\eta| = 1$ for the eigenvalues of ϕ, since ϕ is an expansion map. It is known that if ϕ is an expansion map of a primitive substitution tiling with FLC, every eigenvalue of ϕ is an algebraic integer (Kenyon & Solomyak, 2010; Kwapisz, 2016). Even for non-FLC cases, we know from the rigidity that the control point set lies in a finitely generated free abelian group L which spans $\mathbb{R}^d$ and $\phi L \subset L$. So all the eigenvalues of ϕ are algebraic integers [Lemma 4.1 of Lee & Solomyak (2008)].

In the case of non-unimodular substitution tilings, there are two parts of spaces for the internal space of a CPS. One is a Euclidean part and the other is a profinite group part. We describe them below.

4.1. An internal space for a CPS

4.1.1. Euclidean part for the internal space. In this subsection, we assume that there exists at least one algebraic conjugate whose absolute value is less than 1, which is different from the eigenvalues of ϕ. In the case of unimodular ϕ, we can observe that there always exists such an algebraic conjugate. But in the case of non-unimodular ϕ, it is possible not to have an algebraic conjugate whose absolute value is less than 1. For example, let us consider an expansion map

$$\phi = \begin{pmatrix} 3 + 2^{1/2} & 0 \\ 0 & 3 - 2^{1/2} \end{pmatrix}.$$

Then the minimal polynomial of ϕ is $x^2 - 6x + 7 = 0$, which means that ϕ is non-unimodular. If there exists no other algebraic conjugate of the eigenvalues of ϕ whose absolute value is less than 1, one can skip this subsection and go to the next Section 4.1.2.

Recall that J is the multiplicity of the eigenvalues of ϕ, d is the dimension of the space $\mathbb{R}^d$, m is the number of distinct eigenvalues of ϕ and $d = mJ$. We can write

$$\phi = \begin{bmatrix} \psi_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & \psi_J \end{bmatrix} \text{ and } \psi_j = \psi := \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{s+t} \end{bmatrix} \tag{18}$$

where A_k is a real 1×1 matrix for $1 \leq k \leq s$, a real 2×2 matrix of the form

$$\begin{bmatrix} a_k & -b_k \\ b_k & a_k \end{bmatrix}$$

for $s + 1 \leq k \leq s + t$ with $s, t \in \mathbb{Z}_{\geq 0}$ and $m = s + 2t$. Here O is the $m \times m$ zero matrix and $1 \leq j \leq J$. Then the eigenvalues of ψ are

$$\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \overline{\lambda_{s+1}}, \ldots, \lambda_{s+t}, \overline{\lambda_{s+t}}. \tag{19}$$

Note that m is the degree of the characteristic polynomial of ψ.

We assume that the minimal polynomial of ψ over $\mathbb{Z}$ has e real roots and f pairs of complex conjugate roots. Since the minimal polynomial of ψ has the characteristic polynomial of ψ as a divisor, we can consider the roots of the minimal polynomial of ψ over $\mathbb{Z}$ in the following order:

$$\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \ldots \lambda_e, \lambda_{e+1}, \overline{\lambda_{e+1}}, \ldots, \\ \lambda_{e+t}, \overline{\lambda_{e+t}}, \lambda_{e+t+1}, \overline{\lambda_{e+t+1}}, \ldots, \lambda_{e+f}, \overline{\lambda_{e+f}}.$$

Let

$$n := e + 2f. \tag{20}$$

We now consider a Euclidean space whose dimension is $n - m$, whose number corresponds to the number of the other roots of the minimal polynomial of ψ which are not the eigenvalues of ψ. Let

$$\mathbb{H}_j := \mathbb{R}^{n-m}, \quad 1 \le j \le J.$$

For $1 \le j \le J$, define a $(n - m) \times (n - m)$ matrix

$$D_j := \begin{bmatrix} A_{s+1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & A_e & 0 & \cdots & 0 \\ 0 & \cdots & 0 & A_{e+t+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & A_{e+f} \end{bmatrix}$$

where A_{s+g} is a real 1×1 matrix with the value λ_{s+g} for $1 \le g \le e - s$, and A_{e+t+h} is a real 2×2 matrix of the form

$$\begin{bmatrix} a_{e+t+h} & -b_{e+t+h} \\ b_{e+t+h} & a_{e+t+h} \end{bmatrix}$$

for $1 \le h \le f - t$ [see Lee, Akiyama & Lee (2020) for more details]. The matrix D_j operates on the space $\mathbb{H}_j$.

Notice that ϕ and ψ have the same minimal polynomial over $\mathbb{Z}$, since ϕ is the diagonal matrix containing J copies of ψ.

Let us consider now the following embeddings:

$$\Psi_j : \mathbb{Z}[\phi]\boldsymbol{a}_j \to \mathbb{H}_j, \\ P_j(\phi)\boldsymbol{a}_j \mapsto P_j(D_j)\boldsymbol{\beta}_j,$$

where $P_j \in \mathbb{Z}[x]$, $\boldsymbol{a}_j$ is as in (13), $\boldsymbol{\beta}_j := (1, \ldots, 1) \in \mathbb{H}_j$ and $1 \le j \le J$. Note that

$$\Psi_j(\phi x) = D_j \Psi_j(x) \quad \text{for any } x \in \mathbb{Z}[\phi]\boldsymbol{a}_j.$$

Let $L = \mathbb{Z}[\phi]\boldsymbol{a}_1 + \cdots + \mathbb{Z}[\phi]\boldsymbol{a}_J$. Note that the minimal polynomial of ϕ is monic, since the eigenvalues of ϕ are all algebraic integers. So $\phi L \subset L$ and

$$\{\boldsymbol{a}_1, \ldots, \phi^{n-1}\boldsymbol{a}_1, \ldots, \boldsymbol{a}_J, \ldots, \phi^{n-1}\boldsymbol{a}_J\} \tag{21}$$

is a basis of L as a free $\mathbb{Z}$-module.

Now, we can define the map

$$\Psi_0 : L \to \mathbb{H}_1 \times \cdots \times \mathbb{H}_J,$$

$$P_1(\phi)\boldsymbol{a}_1 + \cdots + P_J(\phi)\boldsymbol{a}_J \mapsto (P_1(D_1)\boldsymbol{\beta}_1, \ldots, P_J(D_J)\boldsymbol{\beta}_J). \tag{22}$$

Since $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_J$ are linearly independent over $\mathbb{Z}[\phi]$, the map Ψ_0 is well defined. Thus $\Psi_0(\phi x) = D\Psi_0(x)$ where

$$D := \begin{bmatrix} D_1 & & O \\ & \ddots & \\ O & & D_J \end{bmatrix} \tag{23}$$

is a block diagonal $(n - m)J \times (n - m)J$ matrix in which D_j is an $(n - m) \times (n - m)$ matrix, $1 \le j \le J$, and $D_1 = \cdots = D_J$. Let $\mathbb{H} := \mathbb{H}_1 \times \cdots \times \mathbb{H}_J = \mathbb{R}^{(n-m)J}$.

4.1.2. Profinite group part for the internal space.

To make the notation short, denote the basis of L given in (21) by $\mathbf{v}_1, \ldots, \mathbf{v}_n, \ldots, \mathbf{v}_{(n-1)J+1}, \ldots, \mathbf{v}_{nJ}$. Consider a $\mathbb{Z}$-module isomorphism between L and $\mathbb{Z}^{nJ}$

$$\pi : L \to \mathbb{Z}^{nJ}, \quad \pi(\mathbf{v}) = (c_1, \ldots, c_n, \ldots, c_{(n-1)J+1}, \ldots, c_{nJ}) \tag{24}$$

where

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n + \cdots + c_{(n-1)J+1}\mathbf{v}_{(n-1)J+1} + \cdots + c_{nJ}\mathbf{v}_{nJ}.$$

Consider the $(d \times nJ)$ matrix:

$$N = [\mathbf{v}_1, \ldots, \mathbf{v}_{nJ}]. \tag{25}$$

Since L spans $\mathbb{R}^d$ over $\mathbb{R}$, the rank of N is d. Thus $N^{\rm T}\mathbf{x} = \mathbf{0}$ has only the trivial solution, where $N^{\rm T}$ is the transpose of N. From $\phi L \subset L$, we can write, for each $1 \le i \le nJ$,

$$\phi\mathbf{v}_i = \sum_{j=1}^{nJ} a_{ji}\mathbf{v}_j \quad \text{for some } a_{ij} \in \mathbb{Z}.$$

Let

$$M = (a_{ij})_{nJ \times nJ}. \tag{26}$$

Notice that in a special case of $J = 1$, *i.e.* $L = \mathbb{Z}[\phi]\boldsymbol{a}_1$, M is the companion matrix of the minimal polynomial of ϕ over $\mathbb{Z}$. Then

$$\phi N = NM. \tag{27}$$

Note that for any $k \in \mathbb{N}$,

$$\phi^k N = NM^k \tag{28}$$

and

$$M^{\rm T}N^{\rm T} = N^{\rm T}\phi^{\rm T}. \tag{29}$$

Notice also that for any $\mathbf{v} \in L$ and for any $k \in \mathbb{N}$,

$$\pi\phi(\mathbf{v}) = M\pi(\mathbf{v}) \quad \text{and} \quad \pi\phi^k(\mathbf{v}) = M^k\pi(\mathbf{v}). \tag{30}$$

Lemma 4.1. Any eigenvalue of ϕ with multiplicity J becomes also the eigenvalue of M with the same multiplicity J. Furthermore the minimal polynomial of ϕ over $\mathbb{Z}$ is the same as the minimal polynomial of M over $\mathbb{Z}$.

Proof. Let λ be an eigenvalue of ϕ with multiplicity J. Since $\phi^{\rm T}$ and ϕ have the same eigenvalues, λ is an eigenvalue of $\phi^{\rm T}$. Let $\mathbf{x}$ be the corresponding eigenvector of $\phi^{\rm T}$. Then

$$M^{\rm T}(N^{\rm T}\mathbf{x}) = N^{\rm T}(\phi^{\rm T}\mathbf{x}) = N^{\rm T}\lambda\mathbf{x} = \lambda(N^{\rm T}\mathbf{x}).$$

Since $\mathbf{x}$ is nonzero, $N^{\rm T}\mathbf{x}$ is nonzero and so λ is an eigenvalue of $M^{\rm T}$. Thus the eigenvalue λ of ϕ becomes also an eigenvalue of

M. Since ϕ is a diagonal matrix, there are $d(= mJ)$ independent eigenvectors. The images of these vectors under $N^{\rm T}$ are the eigenvectors of $M^{\rm T}$ and linearly independent. Since all the eigenvalues of ϕ are algebraically conjugate with the same multiplicity J, all the eigenvalues of ϕ are also eigenvalues of $M^{\rm T}$ with the same multiplicity J. Thus we note that the set of the eigenvalues of M consists of all the eigenvalues of ϕ and all the other algebraic conjugates of them which are not the eigenvalues of ϕ, and the multiplicity of all the eigenvalues of M is J.

Since ϕ is a diagonal matrix and all the eigenvalues of ϕ are algebraic integers, there exists a minimal polynomial of ϕ over $\mathbb{Z}$. Since M is an integer matrix, there exists a minimal polynomial of M over $\mathbb{Z}$ as well. Let $P(x)$ be the minimal polynomial of ϕ over $\mathbb{Z}$ so that $P(\phi) = 0$ where $P(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, $a_i \in \mathbb{Z}$, and $i \in \{0, 1, \ldots, k-1\}$. Then using (30), for any $\mathbf{v} \in L$,

$$
\begin{aligned}
&\pi(P(\phi)(\mathbf{v}))\\
&= \pi((\phi^k + a_{k-1}\phi^{k-1} + \cdots + a_1\phi + a_0 I_d)(\mathbf{v}))\\
&= \pi\phi^k(\mathbf{v}) + a_{k-1}\pi\phi^{k-1}(\mathbf{v}) + \cdots + a_1\pi\phi(\mathbf{v}) + a_0\pi(\mathbf{v})\\
&= M^k\pi(\mathbf{v}) + a_{k-1}M^{k-1}\pi(\mathbf{v}) + \cdots + a_1 M\pi(\mathbf{v}) + a_0\pi(\mathbf{v})\\
&= (M^k + a_{k-1}M^{k-1} + \cdots + a_1 M + a_0 I_{nJ})\pi(\mathbf{v})\\
&= P(M)\pi(\mathbf{v}). \qquad (31)
\end{aligned}
$$

From (31), $P(M)$ is a zero matrix. On the other hand, we can observe that if $P(x)$ is the minimal polynomial of M over $\mathbb{Z}$, then $P(\phi)$ is a zero matrix as well. Thus the minimal polynomial of ϕ over $\mathbb{Z}$ is the same as the minimal polynomial of M over $\mathbb{Z}$. □

We can observe this property of Lemma 4.1 concretely with Example 5.10.

Let us consider the case that ϕ is non-unimodular, *i.e.* $\phi L \subset L$ but $\phi L \neq L$. Let us denote $\mathbb{Z}^{nJ}$ by $\mathcal{L}$ which is a lattice in $\mathbb{R}^{nJ}$. Then $M\mathcal{L} \subset \mathcal{L}$ but $M\mathcal{L} \neq \mathcal{L}$. We define the M-adic space which is an inverse limit space of $\mathcal{L}/M^k\mathcal{L}$ with $k \in \mathbb{N}$. Note that $M : \mathcal{L} \to \mathcal{L}$ is an injective homomorphism. Observe that $[\mathcal{L} : M\mathcal{L}]$ is non-trivial and finite. We have an inverse limit of an inverse system of discrete finite groups,

$$
\begin{aligned}
\overleftarrow{\mathcal{L}}_M &:= \varprojlim \mathcal{L}/M^k\mathcal{L}\\
&= \mathcal{L}/M\mathcal{L} \leftarrow \mathcal{L}/M^2\mathcal{L} \leftarrow \mathcal{L}/M^3\mathcal{L} \leftarrow \cdots\\
&= \{(\mathbf{x}_1 + M\mathcal{L}, \mathbf{x}_2 + M^2\mathcal{L}, \mathbf{x}_3 + M^3\mathcal{L}, \ldots) \mid\\
&\qquad \mathbf{x}_1 \in \mathcal{L}, \mathbf{x}_k \in \mathbf{x}_{k-1} + M^{k-1}\mathcal{L} \quad \text{for each integer } k \geq 2\}, \qquad (32)
\end{aligned}
$$

which is a profinite group. Note that $\overleftarrow{\mathcal{L}}_M$ can be supplied with the usual topology of a profinite group. Note that for any element $\mathbf{x} = (\mathbf{x}_1 + M\mathcal{L}, \mathbf{x}_2 + M^2\mathcal{L}, \mathbf{x}_3 + M^3\mathcal{L}, \ldots) \in \overleftarrow{\mathcal{L}}_M$,

$$M\mathbf{x} = (0 + M\mathcal{L}, M\mathbf{x}_1 + M^2\mathcal{L}, M\mathbf{x}_2 + M^3\mathcal{L}, \ldots).$$

Thus it becomes a compact group which is invariant under the action of M. In particular, the cosets $\mathbf{x} + M^k\overleftarrow{\mathcal{L}}_M$, $\mathbf{x} \in \mathcal{L}$, $k = 0, 1, 2, \ldots$ form a basis of open sets in $\overleftarrow{\mathcal{L}}_M$ and each of these cosets is both open and closed. An important observation is that any two cosets in $\overleftarrow{\mathcal{L}}_M$ are either disjoint or one is contained in the other.

We let ρ denote the Haar measure on $\overleftarrow{\mathcal{L}}_M$, normalized so that $\rho(\overleftarrow{\mathcal{L}}_M) = 1$. Thus for a coset $\mathbf{x} + M^k\overleftarrow{\mathcal{L}}_M$,

$$\rho(\mathbf{x} + M^k\overleftarrow{\mathcal{L}}_M) = \frac{1}{|\det M|^k}.$$

We define the translation-invariant metric d on $\overleftarrow{\mathcal{L}}_M$ via

$$d(\mathbf{x}, \mathbf{y}) := \frac{1}{|\det M|^k} \quad \text{if} \quad \mathbf{x} - \mathbf{y} \in M^k\overleftarrow{\mathcal{L}}_M \setminus M^{k+1}\overleftarrow{\mathcal{L}}_M.$$

Note that $\overleftarrow{\mathcal{L}}_M$ contains a canonical copy of $\mathcal{L}$ via the mapping

$$\iota : \mathcal{L} \to \overleftarrow{\mathcal{L}}_M \text{ such that } \mathbf{x} \mapsto (\mathbf{x} + M\mathcal{L}, \mathbf{x} + M^2\mathcal{L}, \mathbf{x} + M^3\mathcal{L}, \ldots). \qquad (33)$$

We can observe that

$$\iota(M\mathbf{x}) = M(\iota(\mathbf{x})) \quad \text{for } \mathbf{x} \in \mathcal{L}. \qquad (34)$$

Note that $\cap_{k=0}^{\infty} M^k\mathcal{L} = \{0\}$. So we can conclude that the mapping $\mathbf{x} \mapsto \{\mathbf{x} \mod M^k\mathcal{L}\}_k$ embeds $\mathcal{L}$ in $\overleftarrow{\mathcal{L}}_M$. We identify $\mathcal{L}$ with its image in $\overleftarrow{\mathcal{L}}_M$. Note that $\overleftarrow{\mathcal{L}}_M$ is the closure of $\mathcal{L}$ with respect to the topology induced by the metric d.

In the unimodularity case of ϕ, $\phi L = L$ and so $M\mathcal{L} = \mathcal{L}$. Thus $\overleftarrow{\mathcal{L}}_M$ is trivial.

4.2. Concrete construction of a CPS

We construct a CPS taking $\mathbb{R}^d$ as a physical space and $\mathbb{H} \times \overleftarrow{\mathcal{L}}_M$ as an internal space. We will consider this construction dividing ϕ into three cases as given in the following remark. The following construction of a CPS has already appeared in the work of Minervino & Thuswaldner (2014) in the case of $d = 1$. Here we construct a CPS for the case of $d \geq 1$.

Remark 4.2. For an expansion map ϕ, there are three cases.

(i) If ϕ is unimodular, there exists at least one algebraic conjugate λ other than the eigenvalues of ϕ for which $|\lambda| < 1$. Then the map ι in (33) is a trivial map and the internal space is constructed mainly by the Euclidean space discussed in Section 4.1.1.

(ii) If ϕ is non-unimodular and there exists no other algebraic conjugate of the eigenvalues of ϕ whose absolute value is less than 1, then $\mathbb{H}$ is a trivial group and the internal space is constructed exclusively by the profinite group (32) defined in Section 4.1.2.

(iii) If ϕ is non-unimodular and there exist algebraic conjugates (λ's) other than the eigenvalues of ϕ for which $|\lambda| < 1$, then the internal space is a product of the Euclidean space in Section 4.1.1 and the profinite group in Section 4.1.2.

Let us define

$$\Psi : L \to \mathbb{H} \times \overleftarrow{\mathcal{L}}_M \quad \text{for which } \Psi(\mathbf{x}) = (\Psi_0(\mathbf{x}), \iota \circ \pi(\mathbf{x})),$$

where π is defined as in (24). Let us construct a CPS:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{H} \times \overleftarrow{\mathcal{L}}_M & \xrightarrow{\pi_2} & \mathbb{H} \times \overleftarrow{\mathcal{L}}_M \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & \Psi(L) \\ & & \cup & & \\ \mathbf{x} & \longleftarrow\!\shortmid & (\mathbf{x}, \Psi(\mathbf{x})) & \longmapsto & \Psi(\mathbf{x}), \end{array} \tag{35}$$

where π_1 and π_2 are canonical projections,

$$L = \mathbb{Z}[\phi]\boldsymbol{a}_1 + \cdots + \mathbb{Z}[\phi]\boldsymbol{a}_J$$

and

$$\widetilde{L} = \{(\mathbf{x}, \Psi(\mathbf{x})) : \mathbf{x} \in L\}.$$

It is easy to see that $\pi_1|_{\widetilde{L}}$ is injective. We shall show that $\pi_2(\widetilde{L})$ is dense in $\mathbb{H} \times \overleftarrow{\mathcal{L}}_M$ and $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{H} \times \overleftarrow{\mathcal{L}}_M$ in Lemmas 4.3 and 4.4. We note that $\pi_2|_{\widetilde{L}}$ is injective, since Ψ is injective. Since ϕ commutes with the isomorphism σ in Theorem 2.9, we may identify the control point set $\mathcal{C}(\mathcal{T}) = (\mathcal{C}_i)_{i\le\kappa}$ with its isomorphic image. Thus from Theorem 2.9,

$$\operatorname{supp} \mathcal{C}(\mathcal{T}) \subset \mathbb{Z}[\phi]\boldsymbol{a}_1 + \cdots + \mathbb{Z}[\phi]\boldsymbol{a}_J,$$

where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_J \in \mathcal{C}(\mathcal{T})$ and $\operatorname{supp} \mathcal{C}(\mathcal{T}) = \cup_{i\le\kappa}\mathcal{C}_i$. Note that for any $k \in \mathbb{N}$ and $1 \le j \le J$, $\phi^k\boldsymbol{a}_j \in \mathcal{C}(\mathcal{T})$ by the definition of the tile-map. So we can note that

$$L = \langle \cup_{i\le\kappa}\mathcal{C}_i \rangle_{\mathbb{Z}}. \tag{36}$$

Lemma 4.3. $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{H} \times \overleftarrow{\mathcal{L}}_M$.

Proof. For the case (i) of Remark 4.2, $\overleftarrow{\mathcal{L}}_M$ is trivial. So the statement of the lemma follows from Lemma 3.2 of Lee, Akiyama & Lee (2020).

For the case (ii) of Remark 4.2, $\mathbb{H}$ is trivial. Note that the matrix M in (26) is a $d \times d$ integer matrix and L is a lattice in $\mathbb{R}^d$. So $\widetilde{L}$ is a discrete subgroup of $\mathbb{R}^d \times \overleftarrow{\mathcal{L}}_M$ with respect to the product topology. Note that $\mathbb{R}^d \times \overleftarrow{\mathcal{L}}_M \subseteq (L + C_1) \times \overleftarrow{\mathcal{L}}_M \subseteq \widetilde{L} + (C_1 \times \overleftarrow{\mathcal{L}}_M)$, where C_1 is a compact set in $\mathbb{R}^d$. Since $C_1 \times \overleftarrow{\mathcal{L}}_M$ is compact, $\widetilde{L}$ is relatively dense in $\mathbb{R}^d \times \overleftarrow{\mathcal{L}}_M$. Thus the statement of the lemma follows.

For the case (iii) of Remark 4.2, let $L' = \{(\mathbf{x}, \Psi_0(\mathbf{x})) : \mathbf{x} \in L\} \subset \mathbb{R}^d \times \mathbb{H}$. In Lemma 3.2 of Lee, Akiyama & Lee (2020), we notice that the unimodularity property is used only in observing that $\mathbb{H}$ is not trivial in that paper. So by the same argument as Lemma 3.2 of Lee, Akiyama & Lee (2020), we obtain that L' is a lattice in $\mathbb{R}^d \times \mathbb{H}$. This means that L' is a discrete subgroup such that $(\mathbb{R}^d \times \mathbb{H})/L'$ is compact. Notice that $\widetilde{L}$ is still a discrete subgroup in $\mathbb{R}^d \times \mathbb{H} \times \overleftarrow{\mathcal{L}}_M$. Furthermore, $(\mathbb{R}^d \times \mathbb{H} \times \overleftarrow{\mathcal{L}}_M)/\widetilde{L}$ is compact. In fact, note that $\mathbb{R}^d \times \mathbb{H} \subset L' + (C_1 \times C_2)$, where C_1 and C_2 are compact sets in $\mathbb{R}^d$ and $\mathbb{H}$, respectively. Then

$$\mathbb{R}^d \times \mathbb{H} \times \overleftarrow{\mathcal{L}}_M \subset (L' + (C_1 \times C_2)) \times \overleftarrow{\mathcal{L}}_M \subset \widetilde{L} + (C_1 \times C_2 \times \overleftarrow{\mathcal{L}}_M).$$

Since $C_1 \times C_2 \times \overleftarrow{\mathcal{L}}_M$ is compact, $\widetilde{L}$ is relatively dense in $\mathbb{R}^d \times \mathbb{H} \times \overleftarrow{\mathcal{L}}_M$. Thus the statement of the lemma follows. □

Lemma 4.4. $\Psi(L) = \pi_2(\widetilde{L})$ and $\pi_2(\widetilde{L})$ is dense in $\mathbb{H} \times \overleftarrow{\mathcal{L}}_M$.

Proof. For the case (i) of Remark 4.2, $\overleftarrow{\mathcal{L}}_M$ is trivial. So the statement of the lemma follows from Lemma 3.2 of Lee, Akiyama & Lee (2020).

For the case (ii) of Remark 4.2, $\mathbb{H}$ is trivial. Note that $\iota \circ \pi(L) = \mathcal{L}$ and $\mathcal{L}$ is dense in $\overleftarrow{\mathcal{L}}_M$. Thus $\pi_2(\widetilde{L})$ is dense in $\overleftarrow{\mathcal{L}}_M$.

Let us consider the case (iii) of Remark 4.2. It is known from Lee, Akiyama & Lee (2020) that $\Psi_0(L)$ is dense in $\mathbb{H}$. For any open neighbourhood $V \times (\mathbf{z} + M^k\overleftarrow{\mathcal{L}}_M)$ in $\mathbb{H} \times \overleftarrow{\mathcal{L}}_M$, there exists $\mathbf{z}' \in \mathcal{L}$ such that $\iota(\mathbf{z}' + M^\ell\mathcal{L}) \subset \mathbf{z} + M^k\overleftarrow{\mathcal{L}}_M$ for some $\ell \ge k$. Since $\Psi_0(L)$ is dense in $\mathbb{H}$ and $D^\ell\Psi_0(L) = \Psi_0(\phi^\ell L)$, $\Psi_0(\phi^\ell L)$ is dense in $\mathbb{H}$. Note that

$$\Psi_0(\pi^{-1}(\mathbf{z}') + \phi^\ell L) = \Psi_0(\pi^{-1}(\mathbf{z}' + M^\ell\mathcal{L})).$$

So $\Psi_0(\pi^{-1}(\mathbf{z}' + M^\ell\mathcal{L}))$ is dense in $\mathbb{H}$, where π is defined in (24). So

$$\Psi_0(\pi^{-1}(\mathbf{z}' + M^\ell\mathcal{L})) \cap V \neq \emptyset.$$

Hence

$$\Psi(\pi^{-1}(\mathbf{z}' + M^\ell\mathcal{L})) \cap (V \times (\mathbf{z} + M^k\overleftarrow{\mathcal{L}}_M)) \neq \emptyset.$$

Thus $\pi_2(\widetilde{L})$ is dense in $\mathbb{H} \times \overleftarrow{\mathcal{L}}_M$. □

Now that we have proved that (35) is a CPS, we would like to introduce a special projected set $E_{\delta,k}$ which will appear in the proof of the main result in Section 5. For $\delta > 0$ and $k \in \mathbb{Z}_{\ge 0}$, we define

$$\begin{aligned} E_{\delta,k} &:= \wedge(B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^k\overleftarrow{\mathcal{L}}_M) \\ &= \pi_1({\pi_2}^{-1}(\Psi(L) \cap (B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^k\overleftarrow{\mathcal{L}}_M))) \\ &= \{\mathcal{P}(\phi)\boldsymbol{a} \in L : \Psi(\mathcal{P}(\phi)\boldsymbol{a}) \in B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^k\overleftarrow{\mathcal{L}}_M\}, \end{aligned} \tag{37}$$

where $B_\delta^{\mathbb{H}}(\mathbf{0})$ is an open ball around $\mathbf{0}$ with a radius δ in $\mathbb{H}$ and

$$\mathcal{P}(\phi)\boldsymbol{a} = P_1(\phi)\boldsymbol{a}_1 + \cdots + P_J(\phi)\boldsymbol{a}_J \in L.$$

In the following lemma, we find an adequate window for a set $\phi^n E_{\delta,k}$ and note that $E_{\delta,k}$ is a Meyer set.

Lemma 4.5. For any $\delta > 0$ and $k \in \mathbb{Z}_{\ge 0}$, let $E_{\delta,k} = \wedge(B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^k\overleftarrow{\mathcal{L}}_M)$. Then for $n \in \mathbb{N}$,

$$\phi^n E_{\delta,k} = \{\mathcal{Q}(\phi)\boldsymbol{a} \in L : \Psi(\mathcal{Q}(\phi)\boldsymbol{a}) \in (D^n B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{n+k}\overleftarrow{\mathcal{L}}_M\},$$

where $\mathcal{Q}(\phi)\boldsymbol{a} = Q_1(\phi)\boldsymbol{a}_1 + \cdots + Q_J(\phi)\boldsymbol{a}_J \in L$ and $Q_j(\phi) \in \mathbb{Z}[\phi]$ with $1 \le j \le J$. Furthermore $E_{\delta,k}$ forms a Meyer set.

Proof. Note that

$$\begin{aligned}
&\Psi(\mathcal{P}(\phi)\boldsymbol{a}) \in B_\delta(\mathbf{0}) \times M^k \overleftarrow{\mathcal{L}}_M \\
&\Longleftrightarrow (\Psi_0(\mathcal{P}(\phi)\boldsymbol{a}), \iota \circ \pi(\mathcal{P}(\phi)\boldsymbol{a})) \in B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^k \overleftarrow{\mathcal{L}}_M \\
&\Longleftrightarrow (D\ \Psi_0(\mathcal{P}(\phi)\boldsymbol{a}), M\ \iota \circ \pi(\mathcal{P}(\phi)\boldsymbol{a})) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{k+1} \overleftarrow{\mathcal{L}}_M \\
&\Longleftrightarrow (D\ \Psi_0(\mathcal{P}(\phi)\boldsymbol{a}), \iota(M\ \pi(\mathcal{P}(\phi)\boldsymbol{a}))) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{k+1} \overleftarrow{\mathcal{L}}_M \\
&\Longleftrightarrow (\Psi_0(\phi\ \mathcal{P}(\phi)\boldsymbol{a}), \iota(\pi\ \phi(\mathcal{P}(\phi)\boldsymbol{a}))) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{k+1} \overleftarrow{\mathcal{L}}_M \\
&\Longleftrightarrow (\Psi_0(\phi\ \mathcal{P}(\phi)\boldsymbol{a}), \iota \circ \pi(\phi(\mathcal{P}(\phi)\boldsymbol{a}))) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{k+1} \overleftarrow{\mathcal{L}}_M \\
&\Longleftrightarrow \Psi(\phi\ \mathcal{P}(\phi)\boldsymbol{a}) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{k+1} \overleftarrow{\mathcal{L}}_M. \qquad (38)
\end{aligned}$$

The third equivalence comes from (34) and the fourth equivalence comes from (30). Thus

$$\begin{aligned}
\phi E_{\delta,k} &= \phi\{\mathcal{P}(\phi)\boldsymbol{a} \in L : \Psi(\mathcal{P}(\phi)\boldsymbol{a}) \in B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^k \overleftarrow{\mathcal{L}}_M\} \\
&= \{\phi\mathcal{P}(\phi)\boldsymbol{a} \in L : \Psi(\phi\ \mathcal{P}(\phi)\boldsymbol{a}) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{k+1} \overleftarrow{\mathcal{L}}_M\} \\
&= \{\mathcal{Q}(\phi)\boldsymbol{a} \in \phi L : \Psi(\mathcal{Q}(\phi)\boldsymbol{a}) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{k+1} \overleftarrow{\mathcal{L}}_M\} \\
&= \{\mathcal{Q}(\phi)\boldsymbol{a} \in \phi L : (\Psi_0(\mathcal{Q}(\phi)\boldsymbol{a}), \iota(\pi(\mathcal{Q}(\phi)\boldsymbol{a}))) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \\
&\qquad \times M^{k+1} \overleftarrow{\mathcal{L}}_M\} \\
&= \{\mathcal{Q}(\phi)\boldsymbol{a} \in L : \Psi(\mathcal{Q}(\phi)\boldsymbol{a}) \in (D\ B_\delta^{\mathbb{H}}(\mathbf{0})) \times M^{1+k} \overleftarrow{\mathcal{L}}_M\}. \qquad (39)
\end{aligned}$$

In the unimodularity case of ϕ, $\overleftarrow{\mathcal{L}}_M$ is trivial and $\phi L = L$. So the last equality (39) follows. In the non-unimodularity case of ϕ, $\iota(\pi(\mathcal{Q}(\phi)\boldsymbol{a})) \in M^{k+1} \overleftarrow{\mathcal{L}}_M$ implies $\mathcal{Q}(\phi)\boldsymbol{a} \in \phi^{k+1} L$. Since $\phi^{k+1} L \subset \phi L$, $\mathcal{Q}(\phi)\boldsymbol{a} \in \phi L$. This shows the last equality (39). Hence for any $n \in \mathbb{N}$,

$$\phi^n E_{\delta,k} = \{\mathcal{Q}(\phi)\boldsymbol{a} \in L : \Psi(\mathcal{Q}(\phi)\boldsymbol{a}) \in D^n\ B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^{n+k} \overleftarrow{\mathcal{L}}_M\}. \qquad (40)$$

Since (35) is a CPS, $B_\delta^{\mathbb{H}}(\mathbf{0})$ is bounded, and $\overleftarrow{\mathcal{L}}_M$ is compact, $B_\delta^{\mathbb{H}}(\mathbf{0}) \times M^k \overleftarrow{\mathcal{L}}_M$ has a non-empty interior and compact closure, $E_{\delta,k}$ is a model set for each $\delta > 0$ and $k \in \mathbb{Z}_{\geq 0}$. It is given by Moody (1997) and Meyer (1972) that a model set is a Meyer set. Thus $E_{\delta,k}$ forms a Meyer set for each $\delta > 0$ and $k \in \mathbb{Z}_{\geq 0}$. □

5. Main result

Recall that we consider a primitive substitution tiling $\mathcal{T}$ on $\mathbb{R}^d$ with a diagonal expansion map ϕ whose eigenvalues are algebraically conjugate with the same multiplicity J and $\mathcal{T}$ is rigid.

Under the assumption of the rigidity of $\mathcal{T}$, the pure discrete spectrum of $\mathcal{T}$ implies that the set of eigenvalues of ϕ forms a Pisot family [Lemma 5.1 (Lee & Solomyak, 2012)]. Recall that

$$\Xi = \bigcup_{i=1}^{\kappa} (\mathcal{C}_i - \mathcal{C}_i),$$

where $\mathcal{C}(\mathcal{T}) = (\mathcal{C}_i)_{i \leq \kappa}$ is a control point set of $\mathcal{T}$.

Lemma 5.1. Assume that ϕ satisfies the Pisot family condition. Then $\Xi \subset E_{\delta,0}$ for some $\delta > 0$, where $E_{\delta,0}$ is given in (37).

Proof. Notice that the setting for $\mathcal{T}$ fulfils the conditions to use Lemma 4.5 of Lee & Solomyak (2008). So from this lemma, for any $\mathbf{y} \in \Xi$,

$$\mathbf{y} = \sum_{n=0}^{N} \phi^n \mathbf{x}_n, \quad \text{where } \mathbf{x}_n \in U \text{ and } U \text{ is a finite subset in } L.$$

Recall that ϕ is an expansive map and satisfies the Pisot family condition. If there exists at least one algebraic conjugate λ other than the eigenvalues of ϕ for which $|\lambda| < 1$, $\Psi_0(\Xi) \subset B_\delta^{\mathbb{H}}(\mathbf{0})$ for some $\delta > 0$. So $\Psi(\Xi) \subset B_\delta^{\mathbb{H}}(\mathbf{0}) \times \overleftarrow{\mathcal{L}}_M$. If there exists no other algebraic conjugate of the eigenvalues of ϕ whose absolute value is less than 1, $\Psi(\Xi) \subset \{\mathbf{0}\} \times \overleftarrow{\mathcal{L}}_M$. From the definition of $E_{\delta,0}$ in (37), $\Xi \subset E_{\delta,0}$. □

Lemma 5.2. Assume that $\mathcal{T}$ has pure discrete spectrum. Then for any $\mathbf{y} \in L$, there exists $\ell = \ell(\mathbf{y}) \in \mathbb{N}$ such that $\phi^\ell \mathbf{y} \in \Xi$.

Proof. Note from (36) that for any $k \in \mathbb{N}$ and $\boldsymbol{a}_j \in \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_J\}$, $\phi^k \boldsymbol{a}_j$ is contained in Ξ. Recall that $L = \mathbb{Z}[\phi]\boldsymbol{a}_1 + \cdots + \mathbb{Z}[\phi]\boldsymbol{a}_J$. From (10) and (36),

$$\phi(L) \subset \langle \Xi \rangle_{\mathbb{Z}}.$$

So for any $\mathbf{y} \in L$, $\phi\mathbf{y}$ is a linear combination of $\boldsymbol{a}_1, \phi\boldsymbol{a}_1, \ldots, \phi^{n-1}\boldsymbol{a}_1, \ldots, \boldsymbol{a}_J, \phi\boldsymbol{a}_J, \ldots, \phi^{n-1}\boldsymbol{a}_J$ over $\mathbb{Z}$. Applying (11) many times if necessary, we get that for any $\mathbf{y} \in L$, $\phi^\ell y \in \Xi$ for some $\ell = \ell(\mathbf{y}) \in \mathbb{N}$. □

Proposition 5.3. Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with an expansion map ϕ. Under the assumption of the existence of the CPS (35), if $\mathcal{T}$ has pure discrete spectrum, then for any given $\delta > 0$, there exists $K \in \mathbb{N}$ such that

$$\phi^K E_{\delta,0} \subset \Xi. \qquad (41)$$

Proof. Note that $E_{\delta,0}$ is a Meyer set and $\Xi \subset E_{\delta,0}$ for some $\delta > 0$. Since Ξ is relatively dense, for any $\mathbf{x} \in E_{\delta,0}$, there exists $r > 0$ such that $\Xi \cap B_r^{\mathbb{R}^d}(\mathbf{x}) \neq \emptyset$. It is important to note that from the Meyer property of $E_{\delta,0}$, the point set configurations

$$\{\Xi \cap B_r^{\mathbb{R}^d}(\mathbf{x}) : \mathbf{x} \in E_{\delta,0}\}$$

are finite up to translations. Let

$$F = \{\mathbf{u} - \mathbf{x} : \mathbf{u} \in \Xi \cap B_r^{\mathbb{R}^d}(\mathbf{x}) \text{ and } \mathbf{x} \in E_{\delta,0}\}.$$

Then $F \subset L$ and F is a finite set. Thus for any $\mathbf{x} \in E_{\delta,0}$,

$$\mathbf{x} \in \Xi - \mathbf{x} \text{ for some } \mathbf{v} \in F. \qquad (42)$$

From Lemma 5.2, for any $\mathbf{y} \in L$, there exists $\ell = \ell(\mathbf{y}) \in \mathbb{N}$ such that $\phi^{\ell}\mathbf{y} \in \Xi$. Since $\mathcal{T}$ has pure discrete spectrum and so $\mathcal{T}$ admits algebraic coincidence, by (11) there exists $K_1 \in \mathbb{N}$ such that

$$\phi^{K_1}\Xi - \phi^{K_1}\Xi \subset \Xi. \tag{43}$$

Applying the inclusion (43) finitely many times, we obtain that there exists $K_0 \in \mathbb{N}$ such that $\phi^{K_0}F \subset \Xi$. Hence together with (42), there exists $K \in \mathbb{N}$ such that

$$\phi^{K}E_{\delta,0} \subset \Xi. \tag{44}$$

□

Proposition 5.4. Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with a diagonalizable expansion map ϕ whose eigenvalues are algebraic conjugates with the same multiplicity and let $\mathcal{T}$ be rigid. Let Φ be the corresponding κ-set substitution of $\mathcal{T}$ (see Definition 2.3). Suppose that

$$\mathcal{C} = (\mathcal{C}_i)_{i \le \kappa} = \lim_{n\to\infty} (\Phi^N)^n(\{\mathbf{0}\}) \quad \text{and} \quad \phi^N \Xi \subset \mathcal{C}_j \tag{45}$$

for some $\mathbf{0} \in \mathcal{C}_j$, $j \le \kappa$ and $N \in \mathbb{Z}_+$. Then each point set

$$\mathcal{C}_i = \wedge(U_i), \quad i \le \kappa, \tag{46}$$

is a model set in the CPS (35) with a window U_i in $\mathbb{H} \times \overleftarrow{\mathcal{L}_M}$ which is open and precompact.

Proof. For each $i \le \kappa$ and $\mathbf{z} \in \mathcal{C}_i$, there exist $n \in \mathbb{N}$ and $j \le \kappa$ for which

$$\mathbf{z} = f(\mathbf{0}) \text{ with some } f \in (\Phi^n)_{ij}.$$

From $\phi^N \Xi \subset \mathcal{C}_j$,

$$\mathbf{z} + \phi^{n+N}\Xi \subset \mathcal{C}_i.$$

By Theorem 2.7 and Proposition 5.3, there exists $K \in \mathbb{N}$ such that $\phi^K E_{\delta,0} \subset \Xi$. Thus

$$\mathcal{C}_i = \bigcup_{\mathbf{z}\in\mathcal{C}_i} (\mathbf{z} + \phi^{N_{\mathbf{z}}} E_{\delta_{\mathbf{z}},0}),$$

where $N_{\mathbf{z}} \in \mathbb{N}$ and $N_{\mathbf{z}}$ depends on $\mathbf{z}$. Let

$$U_i := \bigcup_{\mathbf{z}\in\mathcal{C}_i} (\mathbf{z}^* + (D^{N_{\mathbf{z}}} B_{\delta_{\mathbf{z}}}(\mathbf{0}) \times M^{N_{\mathbf{z}}} \overleftarrow{\mathcal{L}_M})) \quad \text{for any } i \le \kappa, \tag{47}$$

where $\mathbf{z}^* = \Psi(\mathbf{z})$. Then for any $i \le \kappa$

$$\mathcal{C}_i = \wedge(U_i) \quad \text{where } U_i \text{ is an open set in } \mathbb{H} \times \overleftarrow{\mathcal{L}_M}. \tag{48}$$

In (47), we assume that we have taken the minimal number $N_{\mathbf{z}} \in \mathbb{N}$ so that U_i defined by using $N_{\mathbf{z}} - 1$ does not satisfy (48).

From Lemma 5.1, $\Xi \subset \overleftarrow{E_{\delta,0}}$ for some $\delta > 0$. Thus $\overline{\Psi(\Xi)} \subset \overline{B_\delta^{\mathbb{H}}(\mathbf{0})} \times \overleftarrow{\mathcal{L}_M}$. Since $\overline{B_\delta^{\mathbb{H}}(0)} \times \overleftarrow{\mathcal{L}_M}$ is compact, $\overline{\Psi(\Xi)}$ is compact. Thus $\overline{\Psi(\mathcal{C}_i)}$ is compact. □

Recall from Lagarias & Wang (2003) and Lee *et al.* (2003) that there exists a finite generating set $\mathbf{P}$ such that

$$\mathcal{C} = \lim_{r\to\infty} \Phi^r(\mathbf{P}). \tag{49}$$

Since $\Psi(L)$ is dense in $\mathbb{H} \times \overleftarrow{\mathcal{L}_M}$ by Lemma 4.4, we have a unique extension of Φ to a κ-set substitution on $\mathbb{H} \times \overleftarrow{\mathcal{L}_M}$ in the following way; if $f \in \Phi_{ij}$ for which

$$f : L \to L, \; f(\mathbf{x}) = \phi\mathbf{x} + \mathbf{a},$$

we define

$$f^* : \Psi(L) \to \Psi(L), \; f^*(\mathbf{u}, \mathbf{v}) = (D\mathbf{u}, M\mathbf{v}) + \mathbf{a}^*,$$

where $(\mathbf{u}, \mathbf{v}) \in \Psi(L) \subset \mathbb{H} \times \overleftarrow{\mathcal{L}_M}$, D and M are given in (23) and (26), and $\mathbf{a}^* = \Psi(\mathbf{a})$. If there is no confusion, we will use the same notation f^* for the extended map.

Note that, by the Pisot family condition on ϕ, if there exists at least one algebraic conjugate λ other than the eigenvalues of ϕ for which $|\lambda| < 1$, there exists some $c < 1$ such that $|D\mathbf{x}| \le c \cdot |\mathbf{x}|$ for any $\mathbf{x} \in \mathbb{H}$. Furthermore, from (33)

$$||M\mathbf{x}|| = \frac{1}{|\det M|} ||\mathbf{x}|| \quad \text{for any } \mathbf{x} \in \overleftarrow{\mathcal{L}_M}. \tag{50}$$

By the same argument as in Section 3 of Lee & Moody (2001), the κ-set substitution $\overleftarrow{\Phi}$ induces a multi-component iterated function system on $\overleftarrow{\mathcal{L}_M}$. Thus the κ-set substitution Φ determines a multi-component iterated function system Φ^* on $\mathbb{H} \times \overleftarrow{\mathcal{L}_M}$ and f^* is a contraction on $\mathbb{H} \times \overleftarrow{\mathcal{L}_M}$. Let $S(\Phi^*) = (\mathrm{card}(\Phi^*_{ij}))_{ij}$ be a *substitution matrix* corresponding to Φ^*. Defining the compact subsets

$$W_i = \overline{\Psi(\mathcal{C}_i)} \quad \text{for each } 1 \le i \le \kappa$$

and using (5) and the continuity of the mappings, we have

$$W_i = \bigcup_{j=1}^{\kappa} \bigcup_{f^* \in (\Phi^*)_{ij}} f^*(W_j), \quad i = 1, \ldots, \kappa.$$

This shows that $W_1, \ldots, W_\kappa$ are the unique attractor of Φ^*.

Lemma 5.5. Let

$$U_i := \bigcup_{\mathbf{z}\in\mathcal{C}_i} (\mathbf{z}^* + (D^{N_{\mathbf{z}}} B_{\delta_{\mathbf{z}}}(\mathbf{0}) \times M^{N_{\mathbf{z}}} \overleftarrow{\mathcal{L}_M})) \quad \text{for any } i \le \kappa,$$

where $\mathbf{z}^* = \Psi(\mathbf{z})$, as obtained in (47) with the minimal number $N_{\mathbf{z}}$ satisfying (48). For any $j \le \kappa$ and any $K \in \mathbb{N}$, we have

$$((\Phi^*)^K)_{ij}(U_j) \subset U_i. \tag{51}$$

Proof. For any $i, j \le \kappa$, $(\Phi^K)_{ij}(\mathcal{C}_j) \subset \mathcal{C}_i$. Recall that

$$\mathcal{C}_i = \bigcup_{\mathbf{z}\in\mathcal{C}_i} (\mathbf{z} + \phi^{N_{\mathbf{z}}} E_{\delta_{\mathbf{z}},0}), \quad \text{where } E_{\delta_{\mathbf{z}},0} = \wedge(B_{\delta_{\mathbf{z}}}(\mathbf{0}) \times \overleftarrow{\mathcal{L}_M}).$$

So for any $f \in ((\Phi)^K)_{ij}$, $f(\mathcal{C}_j) \subset \mathcal{C}_i$. Thus for any $f^* \in ((\Phi^*)^K)_{ij}$,

$$f^*(U_j) = \bigcup_{z\in\mathcal{C}_j} (f^*(\mathbf{z}^*) + (D^{N_{\mathbf{z}}+K} B_{\delta_{\mathbf{z}}}(\mathbf{0}) \times M^{N_{\mathbf{z}}+K} \overleftarrow{\mathcal{L}_M})).$$

Notice that $f^*(\mathbf{z}^*) = (f(\mathbf{z}))^*$, where $f(\mathbf{z}) \in \mathcal{C}_i$ and $f \in ((\Phi)^K)_{ij}$. Since we have taken the minimal number $N_\mathbf{z}$ in (47) for each U_i, $f^*(U_j) \subset U_i$. Thus

$$((\Phi^*)^K)_{ij}(U_j) \subset U_i.$$

□

The following proposition shows that the Haar measure of ∂U_i is zero for each $1 \le i \le \kappa$. This is proved using Keesling's argument (Keesling, 1999).

Proposition 5.6. Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with a diagonalizable expansion map ϕ whose eigenvalues are algebraic conjugates with the same multiplicity and let $\mathcal{T}$ be rigid. Let Φ be the corresponding κ-set substitution of $\mathcal{T}$ (see Definition 2.3). If

$$\mathcal{C} = \lim_{n\to\infty} (\Phi^N)^n(\{\mathbf{0}\}) \quad \text{and} \quad \phi^N \Xi \subset \mathcal{C}_j \tag{52}$$

where $\mathbf{0} \in \mathcal{C}_j$, $j \le \kappa$ and $N \in \mathbb{Z}_+$, then each model set $\mathcal{C}_j$, $1 \le j \le \kappa$, has a window with boundary measure zero in the internal space $\mathbb{H} \times \overleftarrow{\mathcal{L}}_M$ of CPS (35).

Proof. Let us define $W_i = \overline{U_i}$, where U_i is the maximal open set in $\mathbb{H} \times \overleftarrow{\mathcal{L}}_M$ satisfying (46). From the assumption of (52), we first note that ϕ fulfils the Pisot family condition from Theorem 2.7 and Lemma 5.1 of Lee & Solomyak (2012). For every measurable set $E \times F \subset \mathbb{H} \times \overleftarrow{\mathcal{L}}_M$ and for any $f^* \in (\Phi^*)_{ij}$ with $f^*((\mathbf{u},\mathbf{v})) = (D\mathbf{u}, M\mathbf{v}) + \mathbf{a}^*$,

$$\eta(f^*(E \times F)) = \eta(D(E) \times M(F) + \mathbf{a}^*) = \frac{|\det D|}{|\det M|}\mu(E)\rho(F),$$

where μ is a Haar meaure in $\mathbb{H}$, ρ is a Haar measure in $\overleftarrow{\mathcal{L}}_M$, $\eta = \mu \times \rho$. Note that $|\det D|/|\det M| < 1$. In particular,

$$\eta(f^*(W_j)) = \frac{|\det D|}{|\det M|}\eta(W_j), \quad 1 \le j \le \kappa,$$

where

$$W_i = \bigcup_{j=1}^{\kappa} \bigcup_{f^* \in (\Phi^*)_{ij}} f^*(W_j).$$

Let us denote $w_j = \mu(W_j)$ for $1 \le j \le \kappa$ and $\mathbf{w} = [w_1, \ldots, w_\kappa]^{\mathrm{T}}$. Then for any $r \in \mathbb{N}$,

$$w_i \le \sum_{j=1}^{\kappa} \left(\frac{|\det D|}{|\det M|}\right)^r \operatorname{card}((\Phi^*)^r)_{ij} w_j.$$

From Proposition 5.3, we know that $w_j > 0$ for any $1 \le j \le \kappa$. Thus

$$\mathbf{w} \le \left(\frac{|\det D|}{|\det M|}\right)^r S((\Phi^*)^r)\mathbf{w} \le \left(\frac{|\det D|}{|\det M|}\right)^r (S(\Phi^*))^r \mathbf{w}$$

for any $r \in \mathbb{N}$.

Note that the Perron–Frobenius eigenvalue of $(S(\Phi^*))^r$ is $|\det\phi|^r$ from Lagarias & Wang (2003). Since the minimal polynomials of ϕ and M over $\mathbb{Z}$ are the same from (27) and the multiplicities of eigenvalues of ϕ and M are the same from Lemma 4.1, we have

$$|\det\phi| \cdot |\det D| = |\det M|.$$

Since $(S(\Phi^*))^r$ is a non-negative primitive matrix with Perron–Frobenius eigenvalue $|\det\phi|$, from Lemma 1 of Lee & Moody (2001)

$$\mathbf{w} = |\det\phi|^r S((\Phi^*)^r)\mathbf{w} = |\det\phi|^r (S(\Phi^*))^r \mathbf{w} \quad \text{for any } r \in \mathbb{N}.$$

By the positivity of $\mathbf{w}$ and $S((\Phi^*)^r) \le (S(\Phi^*))^r$, $S((\Phi^*)^r) = (S(\Phi^*))^r$.

Recall that for any $r \in \mathbb{N}$,

$$W_i = \bigcup_{j=1}^{\kappa} ((\Phi^*)^r)_{ij} W_j. \tag{53}$$

From (3), for any $r \in \mathbb{N}$,

$$(\mathcal{D}^r)_{ij} = \bigcup_{k_1,k_2,\ldots,k_{(r-1)} \le \kappa} (\mathcal{D}_{ik_1} + \phi\mathcal{D}_{k_1k_2} + \cdots + \phi^{r-1}\mathcal{D}_{k_{r-1}j})$$

and

$$(\Phi^r)_{ij}(\{\mathbf{x}_j\}) = \phi^r \mathbf{x}_j + (\mathcal{D}^r)_{ij} \quad \text{for any } \mathbf{x}_j \in \mathcal{C}_j.$$

Note that $W_i = \overline{U_i} = \overline{\Psi(\mathcal{C}_i)}$ and U_i is a non-empty open set. As $r \to \infty$, $\cup_{j=1}^{\kappa} (\mathcal{D}^r)_{ij}$ is dense in W_i. We can find a non-empty open set $V \subset \mathbb{H} \times \overleftarrow{\mathcal{L}}_M$ such that $V \subset \overline{V} \subset U_i$. So there exists $K \in \mathbb{N}$ such that $\mathbf{a} \in (\mathcal{D}^K)_{ij}$ and

$$\{\mathbf{a}^* + (D^K\mathbf{u}, M^K\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in \Psi(\Xi)\} \subset V \subset \overline{V} \subset U_i.$$

Since $W_j \subset \overline{\Psi(\Xi)}$,

$$\begin{aligned}&\{\mathbf{a}^* + (D^K\mathbf{u}, M^K\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in W_j\}\\ &\quad \subset \{\mathbf{a}^* + (D^K\mathbf{u}, M^K\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in \overline{\Psi(\Xi)}\}.\end{aligned}$$

Thus there exists $g^* \in ((\Phi^*)^K)_{ij}$ such that

$$g^*(W_j) \subset U_i. \tag{54}$$

Hence

$$\begin{aligned}\partial U_i = W_i \backslash U_i &= \bigcup_{j=1}^{\kappa} ((\Phi^*)^K)_{ij}(W_j) \setminus U_i\\ &\subset \bigcup_{j=1}^{\kappa} \big(((\Phi^*)^K)_{ij}(W_j)\big) \setminus \big(((\Phi^*)^K)_{ij}(U_j)\big) \qquad (55)\\ &\subset \bigcup_{j=1}^{\kappa} ((\Phi^*)^K)_{ij}(\partial U_j). \qquad (56)\end{aligned}$$

The inclusion (55) follows from Lemma 5.5. Let

$$b_j = \eta(\partial U_j), \ 1 \le j \le \kappa, \quad \text{and} \quad \mathbf{b} = [b_1, \ldots, b_\kappa]^{\mathrm{T}}.$$

Then

$$\mathbf{b} \le |\det\phi|^K S(\Phi^*)^K \mathbf{b}.$$

Thus from (54), there exists a matrix S' for which

$$\begin{aligned}\mathbf{0} \le \mathbf{b} &\le |\det\phi|^K S'\mathbf{b} \le |\det\phi|^K S((\Phi^*)^K)\mathbf{b}\\ &\le |\det\phi|^K (S(\Phi^*))^K \mathbf{b},\end{aligned}$$

where $S' \leq (S(\Phi^*))^M$ and $S' \neq (S(\Phi^*))^M$. If $\mathbf{b} > \mathbf{0}$, again from Lemma 1 of Lee & Moody (2001), $S' = (S(\Phi^*))^M$. This is a contradiction to (54). Therefore $b_j = 0$ for any $1 \leq j \leq \kappa$. □

The regularity property of model sets is shared for all the elements in $X_{\mathcal{T}}$ (see Schlottmann, 1998; Baake *et al.*, 2007; Lee & Moody, 2006). We state it in the following proposition.

Proposition 5.7. [(Schlottmann, 1998), Proposition 7 (Baake *et al.*, 2007), Proposition 4.4 (Lee & Moody, 2006).] Let $\mathcal{C}$ be a Delone κ-set in $\mathbb{R}^d$ for which $\wedge(V_i^\circ) \subset \mathcal{C}_i \subset \wedge(\overline{V_i})$ where $\overline{V_i}$ is compact and $V_i^\circ \neq \emptyset$ for $i \leq \kappa$ with respect to to some CPS. Then for any $\mathbf{\Gamma} \in X_{\mathcal{C}}$, there exists $(-\mathbf{s}, -\mathbf{h}) \in \mathbb{R}^d \times (\mathbb{H} \times \overleftarrow{\mathcal{L}}_M)$ so that

$$-\mathbf{s} + \wedge(\mathbf{h} + V_i^\circ) \subset \Gamma_i \subset -\mathbf{s} + \wedge(\mathbf{h} + \overline{V_i}) \quad \text{for each } i \leq \kappa.$$

From the assumption of pure discrete spectrum and Remark 5.5 of Lee, Akiyama & Lee (2020), we can observe that the condition (52) is fulfilled in the following theorem.

Theorem 5.8. Let $\mathcal{T}$ be a repetitive primitive substitution tiling on $\mathbb{R}^d$ with a diagonalizable expansion map ϕ whose eigenvalues are algebraic conjugates with the same multiplicity and let $\mathcal{T}$ be rigid. If $\mathcal{T}$ has pure discrete spectrum, then each control point set $\mathcal{C}_j$, $1 \leq j \leq \kappa$, is a regular model set in CPS (35) with an internal space which is a product of a Euclidean space and a profinite group.

Proof. Through Section 4.1, we can construct the CPS (35) whose internal space is a product of a Euclidean space and a profinite group. Since $\mathcal{T}$ has pure discrete spectrum and is repetitive, we can find a substitution tiling $\mathcal{S}$ in $X_{\mathcal{T}}$ such that

$$\mathcal{C}_{\mathcal{S}} = \lim_{n\to\infty} (\Phi^N)^n(\{\mathbf{0}\}) \quad \text{and} \quad \phi^N \Xi \subset (\mathcal{C}_{\mathcal{S}})_j, \tag{57}$$

where $\mathbf{0} \in (\mathcal{C}_{\mathcal{S}})_j$, $j \leq \kappa$ and $N \in \mathbb{Z}_+$. From Propositions 5.3, 5.6 and 5.7, the statement of the theorem follows.

Corollary 5.9. Let $\mathcal{T}$ be a repetitive primitive substitution tiling on $\mathbb{R}^d$ with a diagonalizable expansion map ϕ whose eigenvalues are algebraic conjugates with the same multiplicity and $\mathcal{T}$ be rigid. Then $\mathcal{T}$ has pure discrete spectrum if and only if each control point set $\mathcal{C}_j$, $1 \leq j \leq \kappa$, is a regular model set in CPS (35) with an internal space which is a product of a Euclidean space and a profinite group.

Proof. It is known that regular model sets have pure discrete spectrum in quite a general setting (Schlottmann, 2000). Together with Theorem 5.8, we obtain the equivalence between pure discrete spectrum and a regular model set in substitution tilings. □

Now let us look at an example given by Baake *et al.* (1998).

Example 5.10. We look at the example of non-unimodular substitution tiling which is studied by Baake *et al.* (1998). This example is proven to be a regular model set in the setting of a CPS constructed by Baake *et al.* (1998). This has also been considered by Lee, Akiyama & Lee (2020), but it could only be described as a model set, not a regular model set. Here in the setting of CPS (35), we show that this example gives a regular model set. The substitution matrix of the primitive two-letter substitution

$$a \to aab \quad b \to abab$$

has the Perron–Frobenius eigenvalue $\lambda := 2 + 2^{1/2}$ which is a Pisot number. A geometric substitution tiling arising from this substitution can be obtained by replacing symbols a and b in this sequence by the intervals of length $\ell(a) = 1$ and $\ell(b) = 2^{1/2}$. Then we have the following tile-substitution ω

$$\omega(T_a) = \{T_a, 1 + T_a, 2 + T_b\},$$

$$\omega(T_b) = \{T_a, 1 + T_b, 1 + 2^{1/2} + T_a, 2 + 2^{1/2} + T_b\},$$

where $T_a = ([0, 1], a)$ and $T_b = ([0, 2^{1/2}], b)$. Since $x^2 - 4x + 2 = 0$ is the minimal polynomial of λ over $\mathbb{Q}$ and the constant term of the polynomial is 2, the expansion factor λ is non-unimodular. Then we can construct a repetitive substitution tiling $\mathcal{T}$ using the substitution ω.

From Theorem 2.9, we know that the control point set $\mathcal{C}(\mathcal{T})$ fulfils

$$\mathcal{C}(\mathcal{T}) \subset \mathbb{Z}[\lambda]\alpha, \quad \text{for some nonzero element } \alpha \in \mathbb{R}.$$

Let $L = \mathbb{Z}[\lambda]\alpha$ and $\mathcal{L} = \pi(L)$ as in Section 4.1.2. Since

$$\lambda\alpha = 0 \cdot \alpha + 1 \cdot \lambda\alpha$$

$$\lambda^2\alpha = (4\lambda - 2)\alpha = -2 \cdot \alpha + 4 \cdot \lambda\alpha,$$

we get

$$M = \begin{pmatrix} 0 & -2 \\ 1 & 4 \end{pmatrix}.$$

Recall from (32)

$$\overleftarrow{\mathcal{L}}_M = \lim_{\leftarrow k} \mathcal{L}/M^k\mathcal{L}.$$

Let

$$\begin{aligned} \Psi \ : \ \mathbb{Z}[\lambda]\alpha \quad &\to \quad \mathbb{R} \times \overleftarrow{\mathcal{L}}_M \\ P(\lambda)\alpha \quad &\mapsto \ (P(\overline{\lambda}), \iota \circ \pi(P(\lambda)\alpha)), \end{aligned} \tag{58}$$

where $P(x) \in \mathbb{Z}[x]$, $\overline{\lambda} = 2 - 2^{1/2}$, and $\overleftarrow{\mathcal{L}}_M$ is a M-adic space. Since this substitution tiling is known to have pure discrete spectrum (see Baake *et al.*, 1998), it admits an algebraic coincidence. By Proposition 4.4 of Lee (2007) and rewriting the substitution, if necessary, we know that there exists a substitution tiling $\mathcal{S} \in X_{\mathcal{T}}$ such that $\mathbf{\Gamma} = (\Gamma_i)_{i\leq\kappa} := \mathcal{C}(\mathcal{S})$ and

$$\lambda^N \Xi \subset \Gamma_j \quad \text{for which } 0 \in \Gamma_j, j \leq \kappa \text{ and } N \in \mathbb{Z}_+.$$

Then, by the same argument as in Proposition 5.4,

$$\Gamma_i = \bigcup_{z\in\Gamma_j} (z + \lambda^{N_z} E_{\delta_z,0}),$$

where N_z depends on z and

$$E_{\delta_z,0} = \curlywedge(B_{\delta_z}(0) \times \overleftarrow{\mathcal{L}}_M)$$

for some ball $B_{\delta_z}(0)$ of radius δ_z around 0 in $\mathbb{R}$. Let

$$U_i := \bigcup_{z\in\Gamma_i} (\Psi(z) + \overline{\lambda}^{N_z} B_{\delta_z}(0) \times M^{N_z} \overleftarrow{\mathcal{L}}_M) \quad \text{for any } i \leq \kappa.$$

Thus

$$\Gamma_i = \curlywedge(U_i)$$

where U_i is an open set in $\mathbb{R} \times \overleftarrow{\mathcal{L}}_M$ for any $i \leq \kappa$.

From Proposition 5.7, we can observe that the pure discrete spectrum of $\mathcal{T}$ gives a model set with an open and precompact window in the internal space $\mathbb{R} \times \overleftarrow{\mathcal{L}}_M$ for the control point set $\mathcal{C}(\mathcal{T})$. From Proposition 5.6, the measures of the boundaries of the windows are all zero.

Now let us look at another example of a constant-length substitution tiling in $\mathbb{R}$. This example shows that it is important to start with a control point set satisfying the containment (10).

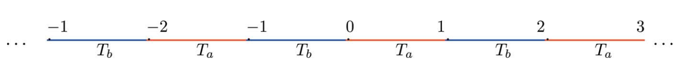
Example 5.11. Consider a two-letter substitution defined as follows:

$$a \to aba \qquad b \to bab.$$

The expansion factor 3 and each prototile can be taken as a unit interval. Starting from $b|a$, we can expand b to the left-hand side and a to the right-hand side, applying the substitution infinite times. Then we get the following bi-infinite sequence:

$$\cdots bababab|abababa \cdots. \tag{59}$$

We consider two prototiles T_a and T_b each of which corresponds to the letter a and the letter b. Following the sequence (59), we replace each letter by the corresponding prototile and obtain a substitution tiling $\mathcal{T}$ which is fixed under the substitution



As a representative point of each tile, if one takes the left end of each interval in the tiling, one gets two point sets Λ_a and Λ_b such that $\Lambda_a = 2\mathbb{Z}$ and $\Lambda_b = 1 + 2\mathbb{Z}$. Since $\Lambda_a \cup \Lambda_b = \mathbb{Z}$, we can take $L = \mathbb{Z}$. Notice in this case that the Euclidean part for the internal space is trivial and the profinite group is

$$\begin{aligned}\overleftarrow{\mathbb{Z}_3} &:= \lim_{\leftarrow k} \mathbb{Z}/3^k\mathbb{Z} \\ &= \mathbb{Z}/3\mathbb{Z} \leftarrow \mathbb{Z}/3^2\mathbb{Z} \leftarrow \mathbb{Z}/3^3\mathbb{Z} \leftarrow \cdots.\end{aligned}$$

Notice that there does not exist $n \in \mathbb{N}$ such that

$$x + 3^n\mathbb{Z} \subseteq 2\mathbb{Z} \quad \text{for some } x \in \mathbb{Z}.$$

This means that neither Λ_a nor Λ_b can be described as a model set projected from a window whose interior is non-empty in $\overleftarrow{\mathbb{Z}_3}$. However the substitution tiling $\mathcal{T}$ has pure discrete spectrum, since it is a periodic structure. The problem here is that the control point set $\boldsymbol{\Lambda} = (\Lambda_i)_{i\in\{a,b\}}$ is not taken to satisfy the containment (10).

On the other hand, if we take the tile map $\gamma : \mathcal{T} \to \mathcal{T}$ for which

$$\gamma(T) = 3x + T_a \quad \text{and} \quad \gamma(S) = 3y + 1 + T_a,$$

where $T = x + T_a$ and $S = y + T_b \in \mathcal{T}$ with $x \in 2\mathbb{Z}$ and $y \in 1 + 2\mathbb{Z}$, then the control point set $\mathcal{C}(\mathcal{T}) = (\mathcal{C}_i)_{i\in\{a,b\}}$ is $\mathcal{C}_a = 2\mathbb{Z}$ and $\mathcal{C}_b = \frac{4}{3} + 2\mathbb{Z}$. So

$$L = \frac{2\mathbb{Z}}{3} \quad \text{and} \quad 3L \subset \Xi,$$

satisfying the containment (10), and the profinite group is

$$\begin{aligned}\overleftarrow{L_3} &:= \lim_{\leftarrow k} L/3^kL \\ &= \left(\frac{2\mathbb{Z}}{3}\right)/2\mathbb{Z} \leftarrow \left(\frac{2\mathbb{Z}}{3}\right)/3\cdot 2\mathbb{Z} \leftarrow \left(\frac{2\mathbb{Z}}{3}\right)/3^2\cdot 2\mathbb{Z} \leftarrow \cdots.\end{aligned}$$

So

$$\mathcal{C}_a = 2\mathbb{Z} = \curlywedge(3\cdot \overleftarrow{L_3}), \mathcal{C}_b = \tfrac{4}{3} + 2\mathbb{Z} = \curlywedge(\tfrac{4}{3} + 3\cdot \overleftarrow{L_3}).$$

Therefore $\mathcal{C}(\mathcal{T})$ can be described as a model set.

6. Further study

In this paper, the rigid structure property of substitution tilings is used to make a connection from pure discrete spectrum to regular model sets, especially to compute the boundary measure of windows. So far, the rigid structure property is known for substitution tilings whose expansion maps (Q) are diagonalizable and the eigenvalues of Q are algebraically conjugate with the same multiplicity (Lee & Solomyak, 2012). Thus it would be useful to know some rigid structure for more general settings. If the rigidity property is precisely known for general substitution tilings, it is expected that we will be able to find the connection from pure discrete spectrum to regular model sets.

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