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Boris Gruber's contributions to mathematical crystallography

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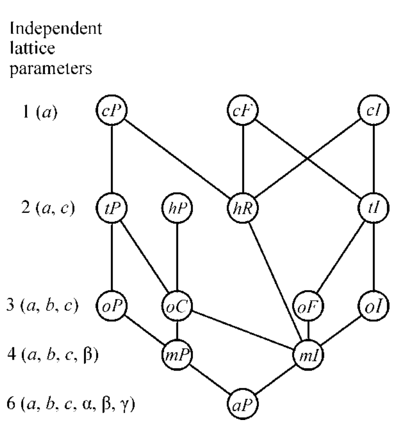
Boris Gruber made fundamental contributions to the study of crystal lattices, leading to a finer classification of lattice types than those of Paul Niggli and Boris Delaunay before him.

1. Introduction

Paul Niggli (Niggli, 1928) used the results of Gotthold Eisenstein (Eisenstein, 1851) on the reduction of ternary positive quadratic forms to define for any given crystal lattice a primitive cell with a unique metric tensor. In this way he obtained 44 lattice characters, which constitute a finer classification of lattices than the 14 Bravais types. Martin J. Buerger considered all primitive cells characterized by the shortest three non-coplanar translations (Buerger, 1957, 1960), known as 'Buerger cells'. Many lattices have more than one Buerger cell, one of which corresponds to Niggli's choice.

Boris Gruber, a Czech mathematician working in Prague at Charles University, became interested in crystallographic lattices and obtained a number of important results, some of which I will mention here. He showed that a given lattice can have up to five Buerger cells (Gruber, 1973), a value reached only by certain lattices of anorthic Bravais type (*aP*). Consider for a Buerger cell the absolute values by which its angles α, β and γ deviate from 90°. Gruber showed that Niggli chose the Buerger cell for which the sum of these three absolute values is maximal (Gruber, 1989). As the number of free parameters characterizing a Buerger cell is largest for *aP* lattices (a, b, c, α, β, γ), it is most important to classify such lattices according to their crystallographic properties. Whereas only two among the 44 lattice characters concern this Bravais type, 43 among Grubers 127 genera correspond to *aP* lattices (Gruber, 1997*a*). This paper was an important step towards the present efforts of dividing crystallographic lattices into a large number of equivalence classes, to which the MACSMIN 2022 meeting in Liverpool, UK, was devoted. Gruber also gave necessary and sufficient conditions satisfied by conventional cells for the 14 Bravais types (see *International Tables for Crystallography*, Volume A, 2002).



2. Boris Gruber and his contributions to mathematical crystallography

Boris Gruber was born in 1921 in the early years of the Czechoslovak Republic. He spent his childhood first in Pilsen then in Brno. He went to a classical grammar school where Latin and Greek were the main subjects. He graduated in 1940, when all Czech universities were closed. To avoid his deportation to national socialist Germany, his parents enrolled him in a technical vocational school. After his second diploma, he joined a factory in Adamov, where his father and younger brother were already employed. During the liberation of Brno, the family went to Bílovice out of fear that their house could be the target of air raids, which indeed happened.

As soon as the war ended, Boris enrolled at Charles University in Prague to study mathematics and physics. His future wife, Zdeňka Kamarýtová, was in the same group as him while he worked on his dissertation. They married in 1951 and had two children, a son Jiří and a daughter Dana.

Boris began his academic career as an assistant to his former professor and later received an offer to teach introductory mathematics courses at the Faculty of Electrical Engineering of the Czech Technical University in Prague. The faculty was located in Poděbrady, a spa resort east of Prague on the river Elbe (Labe in Czech). As he refused to join the communist party, there was no future for him in Poděbrady. He therefore returned in 1958 to the Faculty of Mathematics and Physics in Prague, where he devoted himself to scientific work but was not allowed to teach. His first two publications were in Czech, a paper on the van der Waals equation of state (Gruber, 1950) and a study of the foundations of geometry (Gruber, 1957). Early on, he started using computers and creating programs for them, which led to the papers *Numerical determination of the relative minimum of a function of several variables by quadratic interpolation* (Gruber, 1967) and *On the possibility of applying a computer when solving the four-colour problem* (Gruber, 1971). These publications show that he was able to contribute valuable results to many different fields: physics, pure mathematics and applied mathematics.

However, most by far of Boris Gruber's scientific work concerned crystal lattices. In his early work he developed an algorithm for determining the symmetry and stacking properties of the planes (*hkl*) in the 14 Bravais types of lattices. In 1966 he presented a preliminary account of the algorithm at the 7th Congress of the International Union of Crystallography (IUCr), which was held in Moscow. This research led to the paper *Algorithm for determining the symmetry and stacking properties of the planes (hkl) in a Bravais lattice* (Gruber, 1970*a*). The paper *On the shortest lattice vectors in a three-dimensional translational (Bravais) lattice* appeared in the same year (Gruber, 1970*b*).

Gruber determined the number of Buerger cells for each of the 14 Bravais types (Gruber, 1973): In seven types there is only one Buerger cell, cubic face-centred lattices have 2, in three types there is either 1 or 2, in two types there are 1, 2 or 3 and in the triclinic (anorthic) type there are from 1 to 5. The main results of this paper were obtained during the author's stay at the University of Surrey in the UK, made possible by a grant from the UK Science Research Council.

Křivý and Gruber proposed an algorithm for determining the Niggli cell, starting from an arbitrary primitive cell of a three-dimensional Bravais lattice (Křivý & Gruber, 1976). This algorithm is often used in modern data-reduction software: five among the 100 citations of the paper (up to 22 December 2022) appeared in 2022.

In August 1984, Boris Gruber took part in the Paul Niggli Symposium organized by the Swiss Society for Crystallography. He gave one of the principal lectures, entitled 'Cell reduction' (Gruber, 1984). In this lecture, he stated that the fact that the Niggli cell has no obvious geometrical meaning may look disturbing to physicists and crystallographers. He discussed the Buerger cell with smallest surface, which led in 1989 to his paper *Reduced cells based on extremal principles*. He also mentioned the 24 Delaunay–Voronoi types (also called '*Symmetrische Sorten*') (Delaunay, 1933) which, like Niggli's 44 '*Gitterarten*', are a finer classification of lattices than the 14 Bravais types. The abstract of his lecture is shown in Fig. 1. This line of his thoughts led in 1997 to his paper *Classification of lattices: a new step* (Gruber, 1997*a*).

I had been asked to preside over the session on cell reduction, to which Peter Engel (Bern) and Boris Gruber contributed. We got on well, as shown by the card that he sent me after the symposium (Fig. 2).

In 1989, Gruber succeeded in giving a geometric interpretation of the Niggli choice (Gruber, 1989). He considered the surface S of the cell defined by the vectors **a**, **b** and **c**,

$$\begin{aligned} S &= 2(|\mathbf{b}\times\mathbf{c}| + |\mathbf{c}\times\mathbf{a}| + |\mathbf{a}\times\mathbf{b}|) \\ &= 2(bc\sin\alpha + ca\sin\beta + ab\sin\gamma), \end{aligned}$$

and the deviation W, defined as

$$W = \left|\frac{\pi}{2}-\alpha\right| + \left|\frac{\pi}{2}-\beta\right| + \left|\frac{\pi}{2}-\gamma\right|.$$

Gruber considered the four kinds of Buerger cells for which S or W are minimal or maximal, respectively. He showed that the Niggli cell is the Buerger cell with the largest value of W. Boris Gruber thanked his wife for carefully checking his calculations.

In 1991, de Wolff and Gruber gave an exact definition of Niggli's 44 '*Gitterarten*', which they called 'lattice characters'. In order to represent the characters graphically, they used the projection of the Niggli-reduced basis vector **c** on the **a**, **b** plane (de Wolff & Gruber, 1991).

In his paper *Topological approach to the Niggli lattice characters* (Gruber, 1992) Gruber represented any lattice by a point in $\mathbb{E}^5$, the five-dimensional Euclidean space. He showed that the image of a lattice character is a maximal connected subset of the image of the Bravais type that contains the character. If instead of the Buerger cell with W maximal, another of the four possibilities mentioned above is used to obtain a unique cell, another splitting of Bravais types into lattice characters results. This shows that lattice characters are not as fundamental a concept as the Bravais types.

In 1997 Gruber found a classification of lattices that was finer than Bravais types and of major significance to crystallography. In his paper *Classification of lattices: a new step* (Gruber, 1997*a*), he constructed a classification into 127 classes, called 'genera', which is finer than classifications into Bravais types, Delaunay–Voronoi types (Delaunay, 1933) or lattice characters (see Table 1). Lattices of the same genus agree in the number of Buerger cells, the densest directions and planes, and the symmetry of these planes. Even the formulae for the conventional cells are the same.

Anorthic lattices have six free parameters. Subdividing this Bravais type into a large number of cases according to their crystallographic properties was very welcome. Monoclinic lattices, with four free parameters, were also subdivided into a large number of cases with different crystallographic properties.

The same year, Gruber published another paper, *Alternative formulae for the number of sublattices* (Gruber, 1997*b*). In this paper he considered lattices in n-dimensional Euclidean space and presented formulae for the number of

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2.1 CELL REDUCTION

By B. Gruber, Faculty of Mathematics and Physics, Charles University, Prague, Czechoslovakia.

The problem of the reduced cell may be interpreted as the problem of a unique representation of a Bravais lattice. Since any lattice contains an infinite number of mutually different primitive cells, it seems desirable to choose one of them - the reduced cell - as its representative. The rule of this choice must apply to any lattice and therefore must not depend on the potential symmetry.

The reduced cells are mostly used for determining the symmetry. But it is good to relate also other properties to them (e.g. the closest-packed directions and planes). They answer directly the question of identity or similarity of two lattices. From a more general point of view they are of interest giving an insight into the properties of the three-dimensional space.

From the practical stand-point we want two main things for a reduced cell: an algorithm to find it and a system of conditions (inequalities) to recogzize it. The definition itself is seldom suitable for this purpose being often based on a minimum principle.

1. The Buerger cell is defined by the condition

$$a+b+c = \text{abs min}^{*)} \qquad (1)$$

(Z. Kristallogr. 109, 42, 1957; 113, 52, 1960). Though it is usually not considered a reduced cell being not unique it has many advantages and is often a starting point for other (unique) reduced cells. In 1973 (Acta Cryst. A29, 433) it has been shown that lattices with at most 5 different Buerger cells may exist. But a table listing all ambiguities enables an easy transition between them.

The Buerger cell can be recognized by a system of inequalities and can be achieved by a well working algorithm. It is continuous in the sense that after a sufficiently small but otherwise arbitrary deformation of the lattice one of the Buerger cells of the deformed lattice differs very little from one of the Buerger cells of the original lattice. This is advantageous for determining the symmetry of lattices which are given experimentally.

*) The symbol "abs min" means that we take into account all primitive cells of the lattice whereas "rel min" that we take only those cells which satisfy the preceding condition.

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2. The Niggli cell is the most widely used unique reduced cell nowadays (Handbuch der Experimentalphysik, Vol. 7, Part 1, 1982). It is a particular Buerger cell chosen by a system of conditions which guarantee the uniqueness:

$$a+b+c = \text{abs min}, \quad \text{Niggli conditions} \qquad (2)$$

These conditions have been originally established by Eisenstein for positive definite quadratic forms (J. Math. Crelle 41, 141, 1851). The Niggli cell can be recognized by a system of inequalities and can be directly achieved by an algorithm (Acta Cryst. A32, 297, 1976). It need not be continuous with regard to the deformation of the lattice which causes obvious difficulties for lattices whose parameters were found by measurement. Also the fact that the Niggli cell has no sensible geometrical meaning may be felt disturbing by physicists and crystallographers.

The Niggli cell is chiefly used for determining the symmetry. Niggli in his table has distinguished 44 particular cases, so called characters. De Wolff has recently shown that they can be visualized by means of the projections of the end-point of the vector $\underline{c}$ onto the (a,b)-plane (private communication). The procedure of seeking the symmetry can be somewhat simplified by rearranging the table. Then the actual symmetry is determined by the first entry whose conditions are fulfilled.

3. The Buerger cell with the relatively smallest surface is defined by the requirement

$$a+b+c = \text{abs min}, \quad \text{surface} = \text{rel min} \qquad (3)$$

which is equivalent to the condition

$$a+b+c = \text{abs min}, \ (\underline{a}.\underline{b})^2+(\underline{a}.\underline{c})^2+(\underline{b}.\underline{c})^2 = \text{rel max} \qquad (3a)$$

and also to

$$a+b+c = \text{abs min}, \ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \text{rel max} \qquad (3b)$$

(B. Gruber: Comments on Bravais lattices, microfiches, University of Surrey, England, 1979). The cell is unique but need not have the absolutely smallest surface of all primitive cells. It can be found by means of an algorithm. It is characterized by a system of inequalities which differs only little from the Niggli system. In most lattices both cells coincide but not generally. The Buerger cell with the relatively smallest surface can be used for all purposes as the Niggli cell having moreover the advantage of a simpler definition and an expressive geometrical meaning.

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4. The cells with the absolutely smallest surface,

$$\text{surface} = \text{abs min}, \qquad (4)$$

may be of interest for geometry of physics but are not suitable as reduced cells being not unique. Transferring the concept of the Buerger cell to the reciprocal lattice we ascertain immediately that the cells according to (4) can have even a five-fold ambiguity. In the same way we can get an algorithm how to get them and a system of inequalities which characterize them.

5. The cells with the absolutely smallest surface and relatively shortest edges,

$$\text{surface} = \text{abs min}, \ a+b+c = \text{rel min}, \qquad (5)$$

are unique. They are counter-part of the cells from the point 3. The algorithm and the system of conditions may be gained by means of the reciprocal lattice.

6. The cells with the absolutely shortest edges and absolutely smallest surface,

$$a+b+c = \text{abs min}, \quad \text{surface} = \text{abs min}, \qquad (6)$$

may be of theoretical interest but for our purposes are without meaning since they do not exist in all lattices.

7. The Delaunay's cell or more accurately the Delaunay's quadruplet of vectors $\underline{a}_i$ fulfils

$$\Sigma\, \underline{a}_i = \underline{0}, \quad \underline{a}_i.\underline{a}_j \leq 0 \quad (i \neq j) \qquad (7)$$

any three of the vectors $\underline{a}_i$ forming a basis (Zeit. f. Krist. 84, 132, 1933). Delaunay suggested a way how to find these vectors and use them for determining the symmetry. However, the quatruplet is not unique. We have found an ambiguity but the detailed relationship with the Buerger cells has not yet been established.

As far as the method of our investigations is concerned, almost all assertions are based on an extensive auxiliary table which contains all ambiguities between the Buerger bases (cit. Comments). This table was established chiefly by using the fact that the matrix transforming one Buerger basis into another consists only of the numbers 1, 0, -1.

On the whole we have seen that the requirement of uniqueness is not easily to be fulfilled. All conditions have a twofold character. A simple clear-cut definition would be desirable. The problem of the reduced cell seems still to wait for its final solution.

Figure 1
The abstract of Boris Gruber's lecture at the Paul Niggli Symposium (Gruber, 1984), reproduced with kind permission of the Swiss Society for Crystallography (SSCr/SGK). Note that there is a typographical error in the third line on the second page of this abstract, where 1982 should be 1928.

sublattices of a given index k. They are based on the decomposition of the index k into a product of prime numbers and have the form of a rational function of these primes. Compared with other methods known at the time, they gave the result in a quicker and easier way.

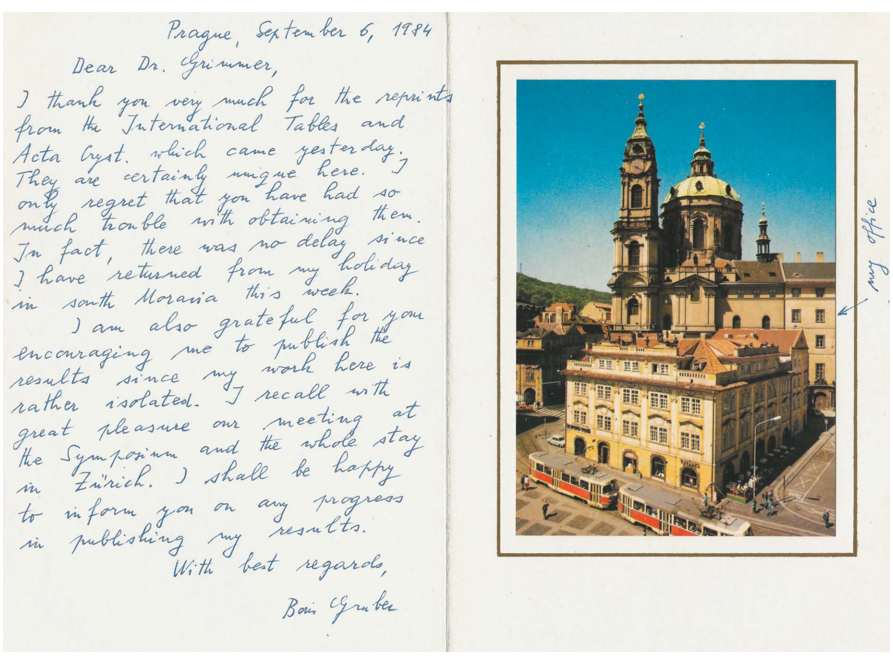
Volume A of *International Tables for Crystallography* (2002) contains a chapter *Further properties of lattices* (Chapter 9.3) by Gruber, where he presents short versions of his results in less technical language, illustrated with three figures. Most important is Section 9.3.4, which deals with conventional cells for the 14 Bravais types. He starts with a remark on Section 9.1.7 written by H. Burzlaff and H. Zimmermann. Gruber shows that the conditions given in Table 9.1.7.2 for a cell to be the conventional cell of a given non-anorthic Bravais type are necessary but not always sufficient. In Table 9.3.4.1 he gives conditions that are necessary and sufficient (Fig. 3). Note that Gruber's notion of a conventional cell differs in several ways from the notion used by Burzlaff and Zimmermann: The orthorhombic cells are chosen such that $a < b$ and $b < c$. A primitive rhombohedral cell is chosen for *hR*, not a rhombohedrally centred hexagonal cell. Gruber takes the centred monoclinic lattice as *I*-centred, not *C*-centred, which simplifies the results. Bravais type *aP* is not mentioned in the table. All lattices that do not satisfy any of the conditions given for the other 13 Bravais types are of type *aP*.

The column 'Conditions' of the table contains footnotes, some of which I give here in abbreviated form. Row *tI*: For $a = c/\sqrt{2}$ the lattice is *cF*. Row *oC*: For $b = a/\sqrt{3}$ the lattice is *hP*. Row *hR*: For $\alpha = 60°$ the lattice is *cF*, for $\alpha = \arccos(-1/3)$ the lattice is *cI*. In row *mI* Gruber gives the conditions for which the lattice is *hR*. Therefore, one may say that *cF* is a limiting case of *tI* as well as of *hR*. In his footnotes Gruber addressed non-trivial limiting cases but did not mention the trivial ones, for instance that *cP* is a limiting case of *tP*. I am convinced that it would have been easy for him to collect all limiting cases and wonder why he did not find it worthwhile.

Table 1
Delaunay–Voronoi types and lattice characters are subdivisions of Bravais types. Genera are a subdivision of the other three classifications.

	Bravais types	Delaunay–Voronoi types	Lattice characters	Genera
cP	1	1	1	**1**
cI	1	1	1	**1**
cF	1	1	1	**1**
hP	1	1	2	**3**
tP	1	2	2	**2**
tI	1	1	4	**5**
hR	1	2	4	**4**
oP	1	1	1	**1**
oI	1	1	3	**7**
oF	1	3	2	**3**
oC	1	1	5	**8**
mP	1	1	3	**5**
mC	1	5	13	**43**
aP	1	3	2	**43**
Total	14	24	44	**127**

In Section 9.3.5 Gruber defined 'conventional characters' as follows: Two lattices of the same Bravais type belong to the same conventional character if and only if one lattice can be deformed into the other in such a way that the conventional parameters of the deformed lattice change continuously from the initial to the final position without change of the Bravais type. The 22 conventional characters are a classification of lattices intermediate between the 14 Bravais types and the 44 lattice characters.


Prague, September 6, 1984

Dear Dr. Grimmer,

I thank you very much for the reprints from the International Tables and Acta Cryst. which came yesterday. They are certainly unique here. I only regret that you have had so much trouble with obtaining them. In fact, there was no delay since I have returned from my holiday in south Moravia this week.

I am also grateful for your encouraging me to publish the results since my work here is rather isolated. I recall with great pleasure our meeting at the Symposium and the whole stay in Zürich. I shall be happy to inform you on any progress in publishing my results.

With best regards,

Boris Gruber

Figure 2
The card that the author received from Boris Gruber after the Paul Niggli Symposium.

Table 9.3.4.1. *Conventional cells*

Bravais type	Centring mode of the cell (**a**, **b**, **c**)	Conditions
cP	*P*	$a = b = c$, $\alpha = \beta = \gamma = 90°$
cI	*I*	$a = b = c$, $\alpha = \beta = \gamma = 90°$
cF	*F*	$a = b = c$, $\alpha = \beta = \gamma = 90°$
tP	*P*	$a = b \neq c$, $\alpha = \beta = \gamma = 90°$
tI	*I*	$c/\sqrt{2} \neq a = b \neq c$,* $\alpha = \beta = \gamma = 90°$
oP	*P*	$a < b < c$,† $\alpha = \beta = \gamma = 90°$
oI	*I*	$a < b < c$, $\alpha = \beta = \gamma = 90°$
oF	*F*	$a < b < c$, $\alpha = \beta = \gamma = 90°$
oC	*C*	$a < b \neq a\sqrt{3}$,‡ $\alpha = \beta = \gamma = 90°$
hP	*P*	$a = b$, $\alpha = \beta = 90°,\ \gamma = 120°$
hR	*P*	$a = b = c$, $\alpha = \beta = \gamma$, $\alpha \neq 60°,\ \alpha \neq 90°,\ \alpha \neq \omega$§
mP	*P*	$-2c\cos\beta < a < c$,¶ $\alpha = \gamma = 90° < \beta$
mI	*I*	$-c\cos\beta < a < c$,** $\alpha = \gamma = 90° < \beta$, (9.3.4.2) but not $a^2 + b^2 = c^2$, $a^2 + ac\cos\beta = b^2$,†† (9.3.4.3) nor $a^2 + b^2 = c^2$, $b^2 + ac\cos\beta = a^2$,‡‡ (9.3.4.4) nor $c^2 + 3b^2 = 9a^2$, $c = -3a\cos\beta$,§§ (9.3.4.5) nor $a^2 + 3b^2 = 9c^2$, $a = -3c\cos\beta$¶¶ (9.3.4.6)

Note: All remaining cases are covered by Bravais type *aP*.

* For $a = c/\sqrt{2}$, the lattice is *cF* with conventional basis vectors $\mathbf{c}, \mathbf{a}+\mathbf{b}, \mathbf{a}-\mathbf{b}$.
† The labelling of the basis vectors according to their length is the reason for unconventional Hermann–Mauguin symbols: for example, the Hermann–Mauguin symbol *Pmna* may be changed to *Pncm*, *Pbmn*, *Pman*, *Pcnm* or *Pnmb*. Analogous facts apply to the *oI*, *oC*, *oF*, *mP* and *mI* Bravais types.
‡ For $b = a\sqrt{3}$, the lattice is *hP* with conventional vectors $\mathbf{a}, (\mathbf{b}-\mathbf{a})/2, \mathbf{c}$.
§ $\omega = \arccos(-1/3) = 109°28'16''$. For $\alpha = 60°$, the lattice is *cF* with conventional vectors $-\mathbf{a}+\mathbf{b}+\mathbf{c},\ \mathbf{a}-\mathbf{b}+\mathbf{c},\ \mathbf{a}+\mathbf{b}-\mathbf{c}$; for $\alpha = \omega$, the lattice is *cI* with conventional vectors $\mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{c}, \mathbf{b}+\mathbf{c}$.
¶ This means that **a**, **c** are shortest non-coplanar lattice vectors in their plane.
** This means that **a**, **c** are shortest non-coplanar lattice vectors in their plane on condition that the cell (**a**, **b**, **c**) is body-centred.
†† If (9.3.4.2) and (9.3.4.3) hold, the lattice is *hR* with conventional vectors $\mathbf{a}, (\mathbf{a}+\mathbf{b}-\mathbf{c})/2, (\mathbf{a}-\mathbf{b}-\mathbf{c})/2$, making the rhombohedral angle smaller than 60°.
‡‡ If (9.3.4.2) and (9.3.4.4) hold, the lattice is *hR* with conventional vectors $\mathbf{a}, (\mathbf{a}+\mathbf{b}+\mathbf{c})/2, (\mathbf{a}-\mathbf{b}+\mathbf{c})/2$, making the rhombohedral angle between 60 and 90°.
§§ If (9.3.4.2) and (9.3.4.5) hold, the lattice is *hR* with conventional vectors $-\mathbf{a}, (\mathbf{a}+\mathbf{b}+\mathbf{c})/2, (\mathbf{a}-\mathbf{b}+\mathbf{c})/2$, making the rhombohedral angle between 90° and ω.
¶¶ If (9.3.4.2) and (9.3.4.6) hold, the lattice is *hR* with conventional vectors $-\mathbf{c}, (\mathbf{a}+\mathbf{b}+\mathbf{c})/2, (\mathbf{a}-\mathbf{b}+\mathbf{c})/2$, making the rhombohedral angle greater than ω.

Figure 3
Table 9.3.4.1, *Conventional cells*, in Volume A of *International Tables for Crystallography* (2002).

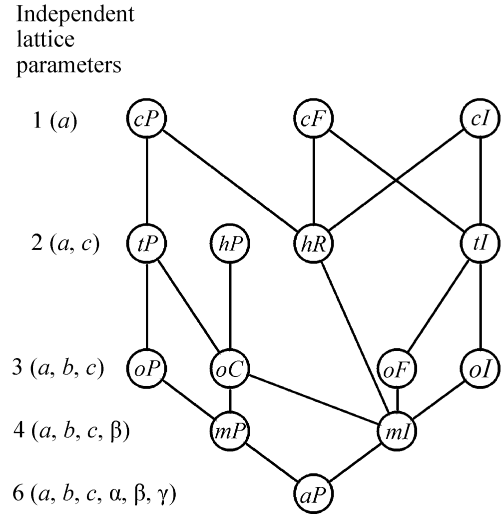


Figure 4
The Bravais type at the upper end of a line is a limiting case of the type at the lower end.

Gruber published his last paper in 2009, many years after retirement. Its title is *On a minimum tetrahedron in a three-dimensional lattice. Part I. Lattices with a shortest basis fulfilling* $\mathbf{b}\cdot\mathbf{c} \geq 0, \mathbf{a}\cdot\mathbf{c} \geq 0, \mathbf{a}\cdot\mathbf{b} \geq 0$ (Gruber, 2009). At the end of the paper, which he devoted to the memory of his wife Zdeňka, he wrote 'The author hopes to find a successor who will complete the whole problem by solving it also in the negative region $a + b + c = \min, \mathbf{b}\cdot\mathbf{c} < 0, \mathbf{a}\cdot\mathbf{c} < 0, \mathbf{a}\cdot\mathbf{b} < 0$'.

In 2015 I published a paper entitled *Partial order among the 14 Bravais types of lattices: basics and applications* (Grimmer, 2015). In this paper I determined the limiting cases of Bravais types and illustrated the results with figures.

I asked Mois Aroyo whether he was interested in including my results in the second online edition of Volume A of *International Tables for Crystallography*, which he was preparing. He suggested extending Table 9.3.4.1 in Volume A of *International Tables for Crystallography* (2002) to show which Bravais types are limiting cases of more general ones. I suggested that Boris Gruber could do this. When Gruber informed Aroyo that his health did not allow him to do so, I took over. In the second online edition of Volume A of *International Tables for Crystallography* (2016) the result is shown in the chapter *Further properties of lattices* (which in this edition is Chapter 3.1.4, by B. Gruber and H. Grimmer). Table 3.1.4.1 shows that the limiting Bravais types are obtained simply by replacing in the column 'Conditions' of the old Table 9.3.4.1 (Fig. 3) any $<$ or $\neq$ symbols by =. The result is illustrated in Fig. 4 (which is Fig. 3.1.4.3 in the chapter).

A photograph of Boris Gruber aged 90 is shown in Fig. 5. He died in 2016.

3. Summary

In his talk at the Paul Niggli Symposium in 1984, Gruber sketched his ideas for a finer classification of lattices that

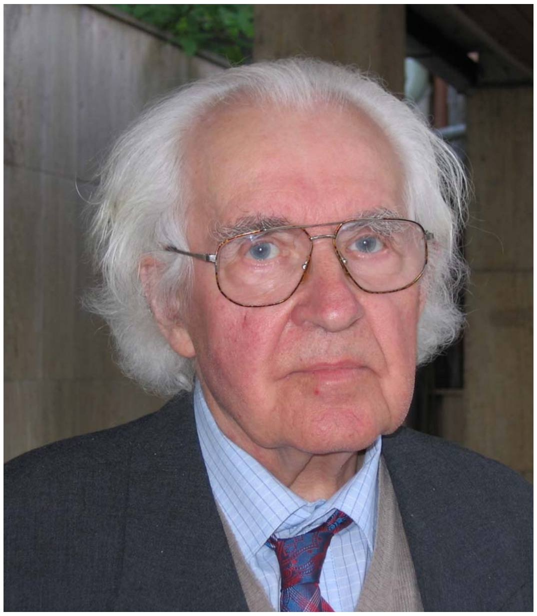


Figure 5
Boris Gruber at age 90. Courtesy of Jiří Gruber.

would subdivide the Bravais types, particularly those of low symmetry, in a way significant to crystallography. This idea culminated in his paper *Classification of lattices: a new step* (Gruber, 1997*a*). This paper is close to the efforts to which the MACSMIN 2022 meeting was devoted, the main difference being that experimental errors in determining the lattice parameters were not considered in the papers of Gruber, as was pointed out by Andrews & Bernstein (2014).

I admire the foresight and perseverance with which Boris Gruber contributed very valuable results on crystal lattices, even though he was not allowed to teach and his international contacts were restricted for a long time.

Acknowledgements

I am very grateful to Jiří Gruber for the photograph shown in Fig. 5 and for information on the life of his father.

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