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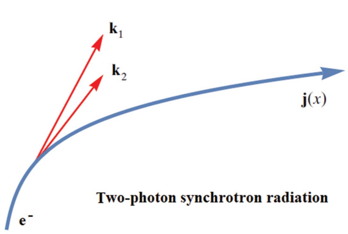

Quantum states of an electromagnetic field interacting with a classical current and their applications to radiation problems

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Synchrotron radiation was originally studied by classical methods using the Liénard–Wiechert potentials of electric currents. Subsequently, quantum corrections to the classical formulas were studied, considering the emission of photons arising from electronic transitions between spectral levels, described in terms of the Dirac equation. In this paper, an intermediate approach is considered, in which electric currents generating the radiation are considered classically while the quantum nature of the radiation is taken into account exactly. Such an approximate approach may be helpful in some cases; it allows one to study one-photon and multi-photon radiation without complicating calculations using corresponding solutions of the Dirac equation. Here, exact quantum states of an electromagnetic field interacting with classical currents are constructed and their properties studied. With their help, the probability of photon emission by classical currents is calculated and relatively simple formulas for one-photon and multi-photon radiation are obtained. Using the specific circular electric current, the corresponding synchrotron radiation is calculated. The relationship between the obtained results and those known before are discussed, for example with the Schott formula, with Schwinger calculations, with one-photon radiation of scalar particles due to transitions between Landau levels, and with some previous results of calculating two-photon synchrotron radiation.

1. Introduction

As a rule, the motion of charged particles in external electromagnetic fields is accompanied by electromagnetic radiation. Important examples, at the same time related to the present work, are synchrotron radiation (SR) and cyclotron (CR) radiation of charged particles in a magnetic field. The phenomenon of SR was discovered approximately 70 years ago (Elder *et al.*, 1947). A large number of works have been devoted to its theoretical description, within the framework of both classical and quantum theory. In both cases, various approximate methods and limiting cases were considered. In classical electrodynamics the electromagnetic field created by an arbitrary electric four-current is described by the Liénard–Wiechert (LW) potentials (Landau & Lifshitz, 1971; Jackson, 1998). It turns out that SR can be described sufficiently precisely in the framework of the classical theory (using LW potentials). Schott was the first to obtain a successful formula for the angular distribution of the power emitted in SR by a particle moving in a circular orbit (Schott, 1907*a*,*b*, 1912). An alternative derivation of the classical formulas describing the properties of SR and their deep analysis, especially for high-energy relativistic electrons, was given by Schwinger

(Schwinger, 1949). Nevertheless, quantum effects may play an important role in SR and CR. In particular, the effects of a back-reaction related to photon radiation, aspects of the discrete structure of the energy levels of electrons in the magnetic field, and the spin properties of charged particles are ignored by classical theory. In this relation, one should mention a new treatment of classical radiative effects, in particular radiation reactions of the electromagnetic field, via effective field theory methods, with an action principle in classical contexts, in particular in the framework of the closed time path formalism (see Birnholtz *et al.*, 2013, 2014). The essence of quantum corrections to classical results was first pointed out by Schwinger (1954). In quantum theory, the radiation rate of the energy of a charge particle in the course of quantum transitions was calculated using exact solutions of the Schrödinger (nonrelativistic case), Klein–Gordon (spinless case) or Dirac (relativistic case) equations with a magnetic field (Sokolov & Ternov, 1957, 1968, 1986). Using his source theory (Schwinger, 1970, 1973*b*), Schwinger had presented an original derivation of similar results (Schwinger, 1973*a*). The quantum treatment revealed a completely new effect of self-polarization of electrons and positrons moving in a uniform and constant magnetic field (Sokolov & Ternov, 1963*a*,*b*). We note that in the latter works only one-photon radiation in the course of quantum transitions was taken into account. However, it has been shown that a multi-photon emission can contribute significantly to the SR (see, for example, Sokolov *et al.*, 1976*a*,*b*). For electromagnetic fields exceeding the critical Schwinger field, $H_0 = m^2c^3/e\hbar$, nonlinear phenomena of quantum electrodynamics begin to play a prominent role. Moreover, at fields comparable with the critical field one can observe nonlinear quantum effects caused by ultrarelativistic particles with high enough momenta. Some examples of such effects [of the orders of α, α^2 (α is a fine-structure constant) in the interaction with the radiation field] are one-photon emission by electrons ($\mathrm{e} \to \mathrm{e}\gamma, \alpha$), pair production by photons ($\gamma \to \mathrm{e}^+\mathrm{e}^-, \alpha$), electron scattering accompanied by pair production ($\mathrm{e} \to \mathrm{e}\mathrm{e}^+\mathrm{e}^-, \alpha^2$), the two-photon emission process ($\mathrm{e} \to \mathrm{e}2\gamma, \alpha^2$) *etc.* If an incident particle has a momentum $p \simeq (H_0/H)m$, then the probabilities of such processes become sufficiently high, and the processes cannot be disregarded.

It should be noted that even the calculation of one-photon radiation using the solutions of the above-mentioned quantum equations is a very complex task. There is an opportunity to simplify these calculations considering in the same relatively simple manner multi-photon radiation taking the quantum nature of the irradiated field into account exactly but considering the particle current classically. This means that we neglect the back-reaction of the radiation to the current that generates this radiation. Such an approximation may be justified in some cases, for example for high-density electron beams. From a technical point of view, this means that for calculating the electromagnetic radiation induced by classical electric currents we have to work with exact quantum states of the electromagnetic field interacting with classical currents. Such an approach is considered in the present work. For these purposes, we first construct exact quantum states of the electromagnetic field interacting with classical currents and study their properties. Then, with their help, we calculate the probability of photon emission by a classical current from the vacuum initial state (*i.e.* from the state without initial photons). Then we obtain relatively simple formulas for one-photon and multi-photon radiation. Using the specific circular electric current we calculate the corresponding SR. We discuss the relationship between the obtained results and those already known, for example with the Schott formula, with Schwinger calculations, with one-photon radiation of scalar particles due to transitions between Landau levels, and with some known results of calculating two-photon SR. Further technical details can be found in Appendices *A*, *B* and *C*.

2. Quantum states of the radiation field interacting with a classical current

Here we consider the quantized electromagnetic field interacting with a classical current $j_\mu(x)$ (see Heitler, 1936; Schweber, 1961; Bogoliubov & Shirkov, 1980*a*; Akhiyeser & Berestetskii, 1981; Gitman & Tyutin, 1986, 1990). In the Coulomb gauge this system is described by a Hamiltonian $\hat{H}$ which consists of two terms, a Hamiltonian of free transversal photons $\hat{H}_\gamma$ and an interaction Hamiltonian $\hat{H}_{\text{int}}$,

$$\hat{H} = \hat{H}_\gamma + \hat{H}_{\text{int}}, \qquad \hat{H}_\gamma = c\hbar \sum_{\lambda=1,2} \int \mathrm{d}\mathbf{k}\, k_0 \hat{c}^\dagger_{\mathbf{k}\lambda}\hat{c}_{\mathbf{k}\lambda},$$
$$\hat{H}_{\text{int}} = \frac{1}{c}\int \left[j_i(x)\hat{A}^i(\mathbf{r}) + \frac{1}{2} j_0(x) A^0(x) \right] \mathrm{d}\mathbf{r}. \tag{1}$$

Here $\hat{A}^i(\mathbf{r})$ are operators (in the Schrödinger representation) of vector potentials of the transversal electromagnetic field,

$$\hat{A}^i(\mathbf{r}) = (4\pi c\hbar)^{1/2} \sum_{\lambda=1}^{2} \int \mathrm{d}\mathbf{k} \left[\hat{c}_{\mathbf{k}\lambda} f^i_{\mathbf{k}\lambda}(\mathbf{r}) + \hat{c}^\dagger_{\mathbf{k}\lambda} f^{i*}_{\mathbf{k}\lambda}(\mathbf{r}) \right],$$
$$i = 1, 2, 3, \tag{2}$$

$$f^i_{\mathbf{k}\lambda}(\mathbf{r}) = \frac{\exp(i\mathbf{k}\mathbf{r})}{\left[2k_0(2\pi)^3\right]^{1/2}} \epsilon^i_{\mathbf{k}\lambda}, \qquad k_0 = |\mathbf{k}|, \tag{3}$$

where $\epsilon^i_{\mathbf{k}\lambda}$ are the polarization vectors of the photon with wavevector $\mathbf{k}$ and polarization $\lambda = 1, 2$. These vectors possess the properties

$$\boldsymbol{\epsilon}_{\mathbf{k}\lambda}\boldsymbol{\epsilon}^*_{\mathbf{k}\sigma} = \delta_{\lambda\sigma}, \quad \boldsymbol{\epsilon}_{\mathbf{k}\lambda}\mathbf{k} = 0, \quad \sum_{\lambda=1}^{2} \epsilon^i_{\mathbf{k}\lambda}\epsilon^{j*}_{\mathbf{k}\lambda} = \delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}. \tag{4}$$

Operators $\hat{c}_{\mathbf{k}\lambda}$ and $\hat{c}^\dagger_{\mathbf{k}\lambda}$ are the annihilation and creation operators of photons with a wavevector $\mathbf{k}$ and polarizations λ. These operators satisfy the commutation relations,

$$\left[\hat{c}_{\mathbf{k}\lambda}, \hat{c}^\dagger_{\mathbf{k}'\lambda'}\right] = \delta_{\lambda\lambda'}\delta(\mathbf{k} - \mathbf{k}'),$$
$$\left[\hat{c}_{\mathbf{k}\lambda}, \hat{c}_{\mathbf{k}'\lambda'}\right] = \left[\hat{c}^\dagger_{\mathbf{k}\lambda}, \hat{c}^\dagger_{\mathbf{k}'\lambda'}\right] = 0. \tag{5}$$

Using equations (2)–(4) one can verify that the operator $\hat{A}^i(\mathbf{r})$ satisfies the condition $\operatorname{div}\hat{\mathbf{A}}(\mathbf{r}) = 0$. We note that in the

Coulomb gauge $A^0(x)$ is a c-valued scalar function which satisfies the following equations,

$$A^0(x) = \int \mathrm{d}\mathbf{r}' \, \frac{j_0(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}, \quad \Delta A^0(x) = -4\pi j_0(x). \tag{6}$$

Then the term $j_0(x)A^0(x)/2$ can be represented as $-2\pi j_0(x)\Delta^{-1} j_0(x)$, and, in the general case, is time dependent.

The evolution of state vectors $|\Psi(t)\rangle$ of the quantized electromagnetic field is governed by the Schrödinger equation,

$$i\hbar\partial_t |\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle. \tag{7}$$

The general solution of equation (7) can be written in the following form (see Bagrov *et al.*, 1974, 1976, 2011),

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle, \tag{8}$$

$$U(t) = \exp\left[-i\hbar^{-1}\hat{H}_\gamma t\right] \exp\left[-i\hbar^{-1}\hat{B}(t)\right],$$

$$\hat{B}(t) = \frac{1}{c}\int_0^t \mathrm{d}t' \int \left\{ j_i(x')\left[\hat{A}^i(x') + \frac{1}{2}\tilde{A}^i(x')\right] + \frac{1}{2}\, j_0(x')A^0(x')\right\} \mathrm{d}\mathbf{r}', \tag{9}$$

$$\tilde{A}^i(x) = \frac{1}{\hbar c}\int_0^t \mathrm{d}t' \int D_0(x - x')\delta_\perp^{ik} j^k(x')\, \mathrm{d}\mathbf{r}',$$

$$\hat{A}^i(x) = (4\pi c\hbar)^{1/2} \sum_{\lambda=1}^{2} \int \mathrm{d}\mathbf{k} \left[\hat{c}_{\mathbf{k}\lambda} f^i_{\mathbf{k}\lambda}(x) + \hat{c}^\dagger_{\mathbf{k}\lambda} f^{i*}_{\mathbf{k}\lambda}(x)\right],$$

$$f^i_{\mathbf{k}\lambda}(x) = f^i_{\mathbf{k}\lambda}(\mathbf{r}) \exp(-ik_0 ct),$$

where $U(t)$ is an evolution operator, and $|\Psi(0)\rangle$ is an initial state of the quantized electromagnetic field at the time instant $t = 0$. The detailed proof that vector (8) is the solution of equation (7) can be found in Appendix A.

The singular function $D_0(x - x')$ can be obtained from the Pauli–Jordan permutation function at $m = 0$ (see, for example, Bogoliubov & Shirkov, 1980*b*),

$$\Box D_0(x - x') = 0,$$

$$D_0(x - x') = 4\pi c\hbar \frac{i}{(2\pi)^3} \int \frac{\mathrm{d}\mathbf{k}}{2k_0} \Big\{ \exp[-ik(x - x')] - \exp[ik(x - x')]\Big\}. \tag{10}$$

It defines nonequal-time commutation relations for the operators $\hat{A}^i(x)$,

$$\left[\hat{A}^i(x), \hat{A}^j(x')\right] = -i\delta_\perp^{ij} D_0(x - x'),$$

$$\delta_\perp^{ij} = \delta^{ij} - \Delta^{-1}\partial^i\partial^j, \tag{11}$$

and is related to the retarded $D^{\mathrm{ret}}(x - x')$ and advanced $D^{\mathrm{adv}}(x - x')$ Green's functions of the D'Alembert equations,

$$\begin{aligned}
\int_0^t \mathrm{d}t'\, D_0(x - x') &= \int_0^\infty \mathrm{d}t'\, D^{\mathrm{ret}}(x - x'),\\
D_0(x - x') &= D^{\mathrm{ret}}(x - x') - D^{\mathrm{adv}}(x - x'),\\
D^{\mathrm{ret}}(x - x') &= \theta(t - t')\, D_0(x - x'),\\
D^{\mathrm{adv}}(x - x') &= \theta(t' - t)\, D_0(x - x'),\\
\Box D^{\mathrm{ret}}(x - x') &= \Box D^{\mathrm{adv}}(x - x') = \delta(x - x').
\end{aligned} \tag{12}$$

Taking into account equations (12), one can see that the functions $\tilde{A}^i(x)$ represent retarded potentials created by a classical current (see, for example, Landau & Lifshitz, 1971; Galtsov *et al.*, 1991).

It is useful to represent the evolution operator $U(t)$ as

$$U(t) = \exp\left[i\phi(t)\right] \exp\left(-i\hbar^{-1}\hat{H}_\gamma t\right) \mathcal{D}(y), \tag{13}$$

$$\mathcal{D}(y) = \exp\left\{ \sum_{\lambda=1}^{2} \int \mathrm{d}\mathbf{k} \left[y_{\mathbf{k}\lambda}(t)\, \hat{c}^\dagger_{\mathbf{k}\lambda} - y^*_{\mathbf{k}\lambda}(t)\, \hat{c}_{\mathbf{k}\lambda}\right]\right\}, \tag{14}$$

$$\phi(t) = -\frac{1}{2c}\int_0^t \mathrm{d}t' \int \left[j_i(x')\, \tilde{A}^i(x') + j_0(x')\, A^0(x')\right] \mathrm{d}\mathbf{r}',$$

$$y_{\mathbf{k}\lambda}(t) = -i\left(\frac{4\pi}{\hbar c}\right)^{1/2} \int_0^t \mathrm{d}t' \int j_i(x')\, f^{i*}_{\mathbf{k}\lambda}(x')\, \mathrm{d}\mathbf{r}'. \tag{15}$$

In the following we omit the argument (t) in functions $y_{\mathbf{k}\lambda}(t)$ to make the formulas more compact.

We recall some basic relations for the displacement operator $\mathcal{D}(\alpha)$ in the Coulomb gauge,

$$\mathcal{D}^\dagger(\alpha) = \mathcal{D}^{-1}(\alpha), \quad |\alpha\rangle = \mathcal{D}(\alpha)|0\rangle, \quad \hat{c}_{\mathbf{k}\lambda}|\alpha\rangle = \alpha_{\mathbf{k}\lambda}|\alpha\rangle, \tag{16}$$

$$\mathcal{D}^\dagger(\alpha)\, \hat{c}_{\mathbf{k}\lambda}\mathcal{D}(\alpha) = \hat{c}_{\mathbf{k}\lambda} + \alpha_{\mathbf{k}\lambda}, \quad \mathcal{D}^\dagger(\alpha)\hat{c}^\dagger_{\mathbf{k}\lambda}\mathcal{D}(\alpha) = \hat{c}^\dagger_{\mathbf{k}\lambda} + \alpha^*_{\mathbf{k}\lambda}.$$

With their help we obtain

$$\mathcal{D}(y)|0\rangle = \exp\left(-\frac{1}{2}\sum_{\lambda=1}^{2} \int \mathrm{d}\mathbf{k}\, |y_{\mathbf{k}\lambda}|^2\right) \times \exp\left(\sum_{\lambda=1}^{2} \int \mathrm{d}\mathbf{k}\; y_{\mathbf{k}\lambda}\hat{c}^\dagger_{\mathbf{k}\lambda}\right)|0\rangle. \tag{17}$$

3. Electromagnetic radiation induced by a classical current

One can use the constructed state vector (8) to study electromagnetic radiation induced by a classical current. For simplicity, we choose the vacuum $|0\rangle$ as the initial state $|\Psi(0)\rangle$ at $t = 0$ in equation (8). The time evolution of this initial state follows from the latter equation,

$$|\Psi(t)\rangle = \exp\left[i\phi(t)\right] \exp\left[-i\hbar^{-1}\, \hat{H}_\gamma t\right] \mathcal{D}(y)|0\rangle. \tag{18}$$

Using equation (18), we can calculate the probability of photon emission.

When operating in a continuous Fock space (see Schweber, 1961), a state with N photons is formed by the repeated action of the photon creation operators on the vacuum $|0\rangle$, and has the form

$$|\{N\}\rangle = (N!)^{-1/2}\prod_{i=1}^{N}\hat{c}^{\dagger}_{\mathbf{k}_i\lambda_i}|0\rangle, \tag{19}$$

where $\hat{c}^{\dagger}_{\mathbf{k}_i\lambda_i}$ are creation operators of photons with wavevector $\mathbf{k}_i$ and polarizations λ_i, $\{N\} = (\mathbf{k}_1\lambda_1, \mathbf{k}_2\lambda_2, \ldots, \mathbf{k}_N\lambda_N)$.

A probability amplitude $R(\{N\},t)$ of the transition from the vacuum state $|0\rangle$ to the state (19) for the time interval t reads

$$R(\{N\},t) = \exp\left[i\phi(t)\right]\langle 0|(N!)^{-1/2}\left(\prod_{i=1}^{N}\hat{c}_{\mathbf{k}_i\lambda_i}\right) \times \exp\left[-i\hat{H}_\gamma t\right]\mathcal{D}(y)|0\rangle. \tag{20}$$

Using properties (16) and (17) of the displacement operator $\mathcal{D}(y)$, and commutation relations (5), one can represent amplitude (20) as follows,

$$\begin{aligned} R(\{N\},t) &= R(0,t)(N!)^{-1/2}\prod_{i=1}^{N}\exp\left[-i|\mathbf{k}_i|ct\right]y_{\mathbf{k}_i\lambda_i}, \\ R(0,t) &= \langle 0|\Psi(t)\rangle \\ &= \exp\left[i\phi(t)\right]\exp\left(-\frac{1}{2}\sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\,|y_{\mathbf{k}\lambda}|^2\right). \end{aligned} \tag{21}$$

Then the corresponding differential probability $P(\{N\},t)$ of such a transition (which we interpret as differential probability of the photon emission) has the form

$$\begin{aligned} P(\{N\},t) &= |R(\{N\},t)|^2 = p(\{N\},t)\,P(0,t), \\ p(\{N\},t) &= (N!)^{-1}\prod_{i=1}^{N}|y_{\mathbf{k}_i\lambda_i}|^2, \\ P(0,t) &= |R(0,t)|^2 = \exp\left(-\sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\,|y_{\mathbf{k}\lambda}|^2\right), \end{aligned} \tag{22}$$

where $P(0,t)$ is thevacuum-to-vacuum transition probability, or the probability of a transition without any photon emission. Thus, $p(\{N\},t)$ is the relative probability of a process in which N photons with quantum numbers $\mathbf{k}_i\lambda_i$ are emitted (the relative differential probability).

One can obtain the total probability $P(N,t)$ of the transition from the vacuum state $|0\rangle$ to the state with N arbitrary photons, summing the quantity $p(\{N\},t)$ over the sets $\{N\}$. Thus, we obtain[1]:

$$\begin{aligned} P(N,t) &= \sum_{\{N\}}P(\{N\},t) = P(0,t)\,p(N,t), \\ \sum_{\{N\}} &= \prod_{i=1}^{N}\left(\sum_{\lambda_i}\int \mathrm{d}\mathbf{k}_i\right), \\ p(N,t) &= (N!)^{-1}\prod_{i=1}^{N}\left(\sum_{\lambda_i}\int \mathrm{d}\mathbf{k}_i\,|y_{\mathbf{k}_i\lambda_i}|^2\right). \end{aligned} \tag{23}$$

[1] It should be noted that Glauber (Glauber, 1951) derived the total probability $P(N,t)$ by his own method, but did not consider its application for the radiation problem.

Introducing a total probability $P(t)$ of the photon emission for the time interval t as follows,

$$\begin{aligned} P(t) &= \sum_{N=1}^{\infty}P(N,t) \\ &= P(0,t)\sum_{N=1}^{\infty}(N!)^{-1}\prod_{i=1}^{N}\left(\sum_{\lambda_i}\int \mathrm{d}\mathbf{k}_i\,|y_{\mathbf{k}_i\lambda_i}|^2\right), \end{aligned} \tag{24}$$

one can easily verify that the relation $P(0,t) + P(t) = 1$ holds true.

The electromagnetic energy of $\{N\}$ photons with given quantum numbers $\{\mathbf{k}\lambda\} = (\mathbf{k}_i\lambda_i,\ i = 1, 2, \ldots, N)$ depends only on their momenta $\{\mathbf{k}\} = (\mathbf{k}_i,\ i = 1, 2, \ldots, N)$ and does not depend on their polarizations; it is equal to

$$W(\{N\}) = \hbar c\left[\sum_{i=1}^{N}|\mathbf{k}_i|\right]. \tag{25}$$

Then the total electromagnetic energy $W(N,t)$ of N emitted photons reads

$$\begin{aligned} W(N,t) &= \sum_{\{N\}}W(\{N\})\,p(\{N\},t) \\ &= \hbar c(N!)^{-1}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\ldots\sum_{\lambda_N=1}^{2} \\ &\quad\times\int \mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2\ldots\mathrm{d}\mathbf{k}_N\left[\sum_{j=1}^{N}|\mathbf{k}_i|\right]\prod_{i=1}^{N}|y_{\mathbf{k}_i\lambda_i}|^2. \end{aligned} \tag{26}$$

It is easy to demonstrate (see Appendix B) that $W(N,t)$ can be represented as

$$\begin{aligned} W(N,t) &= \frac{A}{(N-1)!}\left(\sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\,|y_{\mathbf{k}\lambda}|^2\right)^{N-1}, \\ A &= \hbar c\sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\,k_0|y_{\mathbf{k}\lambda}|^2, \qquad k_0 = |\mathbf{k}|. \end{aligned} \tag{27}$$

Finally, we calculate the total energy $W(t)$ of emitted photons,

$$W(t) = \sum_{N=1}^{\infty}W(N,t). \tag{28}$$

The sum (28) can be calculated exactly, taking into account equation (27),

$$W(t) = A\exp\sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\,|y_{\mathbf{k}\lambda}|^2. \tag{29}$$

4. One-photon radiation by a circular current

Here we study one-photon radiation from the vacuum induced by a specific circular current. Here we are interested in calculating one-photon radiation, so we will discuss the probability of the appearance of one photon with given quantum numbers $\mathbf{k}$ and $\lambda = 1, 2$. Thus, we consider a transition amplitude from the state (18) to the final state of the form (19) with $N = 1$. Using (26), we write one-photon emission as

$$W(1,t) = \hbar c \sum_{\lambda=1}^{2} \int \mathrm{d}\mathbf{k}\, k_0 \left| y_{\mathbf{k}\lambda} \right|^2, \qquad k_0 = |\mathbf{k}|. \tag{30}$$

Let us consider a circular current formed by electrons moving perpendicularly to an external uniform and constant magnetic field $\mathbf{H} = (0, 0, H)$ with the velocity $\mathbf{v}$ along a circular trajectory of radius R. Such a current has the following form (Sokolov & Ternov, 1986),

$$\begin{aligned} &j_0(x) = q\delta^{(3)}[\mathbf{r} - \mathbf{r}(t)], \quad \mathbf{j}(x) = q\dot{\mathbf{r}}(t)\,\delta^{(3)}[\mathbf{r} - \mathbf{r}(t)], \\ &\mathbf{r}(t) = (R\cos\omega t, R\sin\omega t, 0), \\ &\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \omega R(-\sin\omega t, \cos\omega t, 0), \end{aligned} \tag{31}$$

where $q = -e$, $e > 0$ is the electron charge, and $\omega = eH/mc$ is the cyclotron frequency. We disregard the back-reaction of the radiation, *i.e.* we suppose that the current is maintained in its original form during the time interval $\Delta t = t$.

The energy $W(1, t)$ for the current (31) reads

$$\begin{aligned} W(1,t) = {} & \frac{q^2\omega^2}{2\pi} \sum_{n=-\infty}^{+\infty} \int_0^\infty \mathrm{d}k_0 \\ & \times \int_0^\pi \sin\theta\, \mathrm{d}\theta \left[n^2 J_n^2(k_\perp R)\cot^2\theta + k_0^2\, R^2 J_n'^2(k_\perp R) \right] \\ & \times \left| \int_0^t \mathrm{d}t' \exp\left[i(ck_0 - n\omega)t' \right] \right|^2. \end{aligned} \tag{32}$$

The details on calculation of equation (31) can be found in Appendix *C*.

4.1. Derivation of the Schott formula

Let us study the time behaviour of the energy $W(1, t)$ of the one-photon emission (32). One can see that at $t \to \infty$ this quantity as a function of time is not well defined. However, a real physical meaning has the rate $w(t)$ of the energy emission, which is the time derivative of $W(1, t)$,

$$\begin{aligned} w(t) = \partial_t W(1,t) = {} & \frac{q^2\omega^2}{2\pi} \sum_{n=-\infty}^{+\infty} K(t) \int_0^\infty \mathrm{d}k_0 \\ & \times \int_0^\pi \sin\theta \left[n^2 J_n^2(k_\perp R)\cot^2\theta + k_0^2 R^2 J_n'^2(k_\perp R) \right] \mathrm{d}\theta, \\ K(t) = {} & \frac{\partial}{\partial t} \left| \int_0^t \mathrm{d}t' \; \exp\left[i(ck_0 - n\omega)t' \right] \right|^2. \end{aligned} \tag{33}$$

To compare with the Schott result, we have to consider $w(t)$ as $t \to \infty$. In fact the problem is reduced to calculating $\lim_{t\to\infty} K(t)$. This limit can be easily calculated,

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} \frac{2\sin(ck_0 - n\omega)\,t}{ck_0 - n\omega} = 2\pi\delta(ck_0 - n\omega) \tag{34}$$

(see, for example, Sokolov & Ternov, 1986). Taking equation (34) into account and the fact that the delta-function on the right-hand side of equation (34) vanishes for negative n, we obtain

$$\begin{aligned} \lim_{t\to\infty} w(t) = {} & \frac{q^2\omega^2}{c} \sum_{n=1}^{+\infty} n^2 \int_0^\pi \sin\theta \left[J_n^2\left(\frac{n\omega R}{c}\sin\theta \right) \cot^2\theta \right. \\ & \left. + \frac{\omega^2 R^2}{c^2} J_n'^2\left(\frac{n\omega R}{c}\sin\theta \right) \right] \mathrm{d}\theta. \end{aligned} \tag{35}$$

The result (35) reproduces literally the Schott formula for the rate of the energy radiation by a classical current.

4.2. Schwinger calculations of the one-photon radiation

Schwinger (1949) considered classical SR, using a method based on an examination of the energy transfer rate from the electron to the electromagnetic field. Later he calculated the quantum corrections of the first order in $\hbar$ to the classical formula, taking into account the quantum nature of the radiating particle but neglecting its spin properties (Schwinger, 1954). In 1973 he reexamined the problem, utilizing the source theory to obtain the quantum expression for the spectral distribution of the radiated power (Schwinger, 1973*a*).

Schwinger (1949) presented several different distributions of the instantaneous power. Among them was an expression for the power radiated into a unit solid angle about the direction $\mathbf{n} = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$ and contained a unit angular frequency interval about the frequency ck_0,

$$\begin{aligned} P(\mathbf{n}, k_0) = {} & \sum_{n=1}^{\infty} \delta(ck_0 - n\omega)\, P_n(\mathbf{n}), \\ P_n(\mathbf{n}) = {} & \frac{\omega^2 R}{c^2} \frac{q^2}{2\pi} n^2 \left[\frac{\omega^2 R^2}{c^2} J_n'^2\left(\frac{n\omega R}{c}\cos\theta \right) \right. \\ & \left. + \frac{\sin^2\theta}{\cos^2\theta} J_n\left(\frac{n\omega R}{c}\cos\theta \right) \right]. \end{aligned} \tag{36}$$

The total radiated power can be calculated as

$$P = \int_0^\infty c\,\mathrm{d}k_0 \int P(\mathbf{n}, k_0)\,\mathrm{d}\Omega. \tag{37}$$

Considering the high-frequency radiation,

$$1 - \frac{\omega^2 R^2}{c^2} \ll 1, \qquad \theta \ll 1, \quad n \gg 1, \tag{38}$$

and using the connection between the Airy and Bessel functions, Schwinger obtained an alternative representation for his result in the form

$$\begin{aligned} P_n(\mathbf{n}) = {} & \frac{q^2\omega}{6\pi^2 R} n^2 \left(1 - \omega^2 R^2/c^2 + \theta^2 \right)^2 \\ & \times \left[K_{2/3}^2(\zeta) + \frac{\theta^2 K_{1/3}^2(\zeta)}{1 - \omega^2 R^2/c^2 + \theta^2} \right], \\ \zeta = {} & \frac{n}{n_c} \left(\frac{1 - \omega^2 R^2/c^2 + \theta^2}{1 - \omega^2 R^2/c^2} \right)^{3/2}, \end{aligned} \tag{39}$$

and n_c is a critical harmonic number (Schwinger, 1949). Note that a formal difference in angular distribution between (36) and (35) appear due to different notation and does not lead to any differences in the final values.

Schwinger (1954) considered the quantum corrections of the first order in $\hbar$ to the classical formula, taking into account the quantum nature of the radiating electron. He neglected the spin properties as at this level of accuracy the spin degrees of freedom play no role for unpolarized particles. The first-order in the $\hbar$ correction to the classical formula (37) can be obtained from the classical expression for the differential radiation probability $(ck_0)^{-1}P(\mathbf{n}, ck_0)$ (Schwinger, 1954) by making the substitution

$$ck_0 \rightarrow ck_0\left(1+\frac{\hbar ck_0}{E}\right). \tag{40}$$

The total radiated power with the first-order quantum corrections obtained by Schwinger reads

$$w=\frac{2}{3}\omega\frac{q^2}{R}\left(\frac{E}{mc^2}\right)^4\left[1-\sqrt{3}\,\frac{55}{16}\frac{\hbar}{mcR}\left(\frac{E}{mc^2}\right)^2+O\big(\hbar^2\big)\right]. \tag{41}$$

Schwinger (1973*a*) considered the radiation of a spinless charged particle in a homogeneous magnetic field, and obtained the spectral distribution of the radiated power $w(k_0)$ (here $c = \hbar = 1$) in the form

$$\begin{aligned} w(k_0) &= \frac{ck_0q^2}{\pi m}\frac{m^2}{E^2}\left\{\int_0^\infty\frac{\mathrm{d}x}{x}\left(1+2x^2\right)\right.\\ &\quad\left.\times\sin\left[\frac{ck_0}{\omega}\left(\frac{m}{E}\right)^3\left(x-\frac{x^3}{3}\right)\right]-\frac{1}{2}\pi\right\},\\ x &= \frac{1}{2}\,\omega t\,\frac{E}{m}. \end{aligned} \tag{42}$$

According to the author, equation (42) in the classical limit reproduces the Schott formula.

Note that the formulas (43) and (42) include both the corrections due to electron recoil and the effects of quantization of the electromagnetic field. As for the comparison with our result, the angular distributions coincide with the Schott formula and are not affected by quantum corrections.

4.3. One-photon radiation of scalar particles due to transitions between Landau levels

When presenting the results obtained by other authors, we use the same system of units that was utilized in the cited articles.

There is a different approach to calculation of radiation of the spinless charged particle due to one-photon transitions between the energy levels presented by Bordovitsyn (2002) and Bagrov (1965). These calculations are based on the exact solutions of the Klein–Gordon equation in a uniform magnetic field (the Furry picture approach). The spectral angular distribution of the radiated power in this approach has the form

$$\begin{aligned} w &= \frac{27}{16\pi^2}w_0\xi^2\varepsilon_0^{-5/2}\int_0^\infty\mathrm{d}y\int_0^\pi\frac{\sin\theta\,\mathrm{d}\theta}{(1+\xi y)^3}\,y^2\\ &\quad\times\left[\varepsilon^2K_{2/3}^2(z_0)+\varepsilon\cos\theta K_{1/3}^2(z_0)\right], \end{aligned} \tag{43}$$

$$w_0=\frac{8}{27}\frac{q^2m^2c^2}{\hbar^2},\quad \xi=\frac{3}{2}\frac{e\hbar H}{m^2c^3}\frac{E}{mc^2},\quad \varepsilon_0=\left(\frac{mc^2}{E}\right)^2,$$

$$z_0=\frac{y}{2}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{3/2},\quad \varepsilon=1-\frac{\omega^2R^2}{c^2}\sin^2\theta,\quad E=\frac{mc^2}{(1-\omega^2R^2/c^2)^{1/2}},$$

where $K_n(z_0)$ are Airy functions, and E is the electron energy. Unfortunately, no representation of (43) in terms of the Bessel functions is given by the authors; however, it is claimed that equation (43) in the limit $\hbar \rightarrow 0$ reproduces the classical result.

5. Two-photon radiation

The probability $p(2,t)$ and the energy $W(2,t)$ of the two-photon radiation for a circular current (31) have the form

$$\begin{aligned} p(2,t) &= \frac{\alpha^2}{(2\pi)^2}\left\{\int\frac{\mathrm{d}\mathbf{k}}{2k_0}\left[k_0^{-2}F_1(\mathbf{k},t)\cot^2\theta+R^2F_2(\mathbf{k},t)\right]\right\}^2,\\ W(2,t) &= \frac{\alpha^2\hbar c}{(2\pi)^2}\left\{\int\mathrm{d}\mathbf{k}\left[k_0^{-2}F_1(\mathbf{k},t)\cot^2\theta+R^2F_2(\mathbf{k},t)\right]\right\}\\ &\quad\times\left\{\int\frac{\mathrm{d}\mathbf{k}'}{k_0'}\left[k_0'^{-2}F_1(\mathbf{k}',t)\cot^2\theta'+R^2F_2(\mathbf{k}',t)\right]\right\}, \end{aligned} \tag{44}$$

where

$$\begin{aligned} F_1(\mathbf{k},t) &= \left|\sum_{n=-\infty}^{+\infty}nJ_n(k_\perp R)F_\mathbf{k}^n(\varphi,t)\right|^2,\\ F_2(\mathbf{k},t) &= \left|\sum_{n=-\infty}^{+\infty}J_n'(k_\perp R)F_\mathbf{k}^n(\varphi,t)\right|^2. \end{aligned} \tag{45}$$

It is useful to compare our results with the calculations of two-photon radiation presented in other works. Voloshchenko *et al.* (1976) considered the bremsstrahlung of relativistic electrons in the so-called approximation of soft photons (the total energy of emitted photons is much less than the energy of a relativistic electron). Our initial assumption, that the classical current $j(x)$ remains unchanged, despite the radiation losses, matches with this approximation. Voloshchenko *et al.* (1976) had used the expression for the instantaneous spectral distribution of the radiation energy of an electron using the Liénard–Wiechert potentials. In such a way they obtained the total electromagnetic energy of the one-photon radiation. If the electric current in the latter quantity is taken in the form (31), it coincides with our result $W(1,t)$ given by equation (32). Then the probability of emitting a photon is defined by the authors as $p(\{1\},t) = W(\{1\},t)/(\hbar ck_0)$ [here $W(\{1\},t)$ is the integrand of $W(1,t)$] and the probability $p(\{N\},t)$ of emitting $\{N\}$ soft photons in a narrow range of angles along the electron motion direction reads

$$p(\{N\},t)=\prod_{i=1}^{N}p\big(1_{\mathbf{k}_i\lambda_i},t\big)=\prod_{i=1}^{N}\left|y_{\mathbf{k}_i\lambda_i}\right|^2. \tag{46}$$

According to the authors, 'when integrating in a finite interval of frequencies and directions, one must introduce a factor $(N!)^{-1}$ that takes into account the identity of the photons'. Thus, they arrive at our result (22), which contains such a factor for any momenta $\mathbf{k}$ without heuristic prescriptions. It is easy to verify that using the same approximation of the small difference between the angles φ_1 and φ_2 of photons emitted, $\Delta\varphi = (\varphi_1 - \varphi_2) \ll 1$, we obtain from equation (32) for the probability of the two-photon radiation the following result,

$$p(2, t) = \frac{25}{24}\,\alpha^2\omega\gamma\Delta\varphi, \qquad \gamma = \left(1 - \omega^2 R^2/c^2\right)^{-1/2}. \tag{47}$$

It coincides with that of the work (Voloshchenko *et al.*, 1976).

It should be noted that Sokolov *et al.* (1976*a*),*b*) calculated two-photon synchrotron emission considering electron transitions between Landau levels by the help of the corresponding solutions of the Dirac equation. In the approximation accepted in the work (Voloshchenko *et al.*, 1976) they derived corrections to equation (47) of the order $\hbar$ due to the quantum nature of the electron and due to its spin.

6. Concluding remarks

As was noted in the *Introduction*, SR was originally studied by classical methods using the Liénard–Wiechert potentials of electric currents. Subsequently, it became clear that in some cases quantum corrections to classical results may be important. These corrections were studied considering the emission of photons arising from electronic transitions between spectral levels, described in terms of the Dirac equation. In this paper, we have considered an intermediate approach, in which electric currents generating the radiation are treated classically while the quantum nature of the radiation is taken into account exactly. Such an approximate approach allows one to study one-photon and multi-photon radiation without complicating calculations using corresponding solutions of the Dirac equation. We have constructed exact quantum states (8) of the electromagnetic field interacting with classical currents and studied their properties. With their help, we have calculated the probability of photon emission by classical currents from the vacuum initial state and obtained relatively simple general formulas for one-photon and multi-photon radiation. Using the specific circular electric current, we have calculated the corresponding one-photon and two-photon SR. It was demonstrated that the emitted single-photon power per unit time in the limit $t \to \infty$ coincides with the classical expression obtained by Schott. This is not strange, since Schott's result was already semi-classical, since he treated the electromagnetic field in terms of Maxwell's equations. It is well known that (see, for example, Akhiyeser & Berestetskii, 1981), in fact, Maxwell equations can be interpreted as the Schrödinger equation for a single photon; the absence of the Planck constant $\hbar$ in these equations as well as in the Schott formula is associated with the masslessness of the photon. The consideration of the electromagnetic radiation in a semi-classical manner, using Maxwell's equations, often allows one to study quantum effects of radiation (Jaynes & Cummings, 1963). Schwinger's calculations of SR contain $\hbar$ since he used elements of quantum field theory that take into account the quantum character of electron motion and in the limit $\hbar \to 0$ lead to the Schott result. The same situation takes place with calculations of the SR of a spinless charged particle due to transitions between energy levels with one-photon emission presented by Bordovitsyn (2002) and Bagrov (1965). The proposed approach provides an opportunity to separate the effects of radiation associated with the quantum nature of the electromagnetic field from the effects caused by the quantum nature of the electron. The calculation of multiphoton corrections is significantly simplified compared, for example, with the approach described by Sokolov *et al.* (1976*a*,*b*) and Voloshchenko *et al.* (1976), where a two-photon correction to the radiation of an electron moving in a circular orbit in a constant uniform magnetic field is calculated within the framework of the Furry picture. Finally, it becomes possible to study the initial states of the system other than the vacuum initial state (the state without initial photons). Using these state vectors, the probabilities $p(N, t)$ (23) and the energy $W(N, t)$ (27) of N photon radiation induced by classical currents are derived. The latter quantity can be summed exactly representing the total energy $W(t)$ (29) of emitted photons. The obtained results can be used for the systematic study of the multiphoton SR.

APPENDIX *A*
Solution of the Schrödinger equation with classical current

Let us verify directly that state vector (8) satisfies equation (7). Foremost, as the operator $\hat{H}_\gamma$ is time-independent, we have

$$i\hbar\partial_t\left[\exp\left(-i\hbar^{-1}\hat{H}_\gamma t\right)\right] = \hat{H}_\gamma \exp\left(-i\hbar^{-1}\hat{H}_\gamma t\right). \tag{48}$$

However, the derivative $\partial_t \hat{A}^i(x)$ does not commute with the operators $\hat{A}^i(x')$, so when calculating the derivative $i\hbar\partial_t$ of the second exponent in the right-hand side of equation (9) one has to use Feynman's method of disentangling operators (Feynman, 1951). Calculating the derivative $i\hbar\partial_t$ in such a way, we find

$$\begin{aligned} &i\hbar\partial_t \exp\left[-i\hbar^{-1}\hat{B}(t)\right] = \hat{K}(t)\exp\left[-i\hat{B}(t)\right],\\ &\hat{K}(t) = \int_0^1 \mathrm{d}s\, \exp\left[-is\hbar^{-1}\hat{B}(t)\right]\left[\partial_t\hat{B}(t)\right]\exp\left[is\hbar^{-1}\hat{B}(t)\right],\\ &\partial_t\hat{B}(t) = \frac{1}{c}\int\left\{j_i(x)\left[\hat{A}^i(x) + \frac{1}{2}\,\tilde{A}^i(x)\right] + \frac{1}{2}\,j_0(x)A^0(x)\right\}\mathrm{d}\mathbf{r}. \end{aligned} \tag{49}$$

Using the operator relation

$$\exp(\hat{A})\,\hat{M}\exp(-\hat{A}) = \hat{M} + \left[\hat{A}, \hat{M}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{M}\right]\right] + \ \ldots$$

we represent the integrand in the right-hand side of $\hat{K}(t)$ as follows,

$$\begin{aligned} &\exp\left[-is\hbar^{-1}\hat{B}(t)\right]\left[\partial_t\hat{B}(t)\right]\exp\left[is\hbar^{-1}\hat{B}(t)\right] = \\ &\qquad \partial_t\,\hat{B}(t) + \left[-is\hbar^{-1}\hat{B}(t), \partial_t\hat{B}(t)\right] \\ &\qquad + \frac{1}{2!}\left[-is\hbar^{-1}\hat{B}(t), \left[-is\hbar^{-1}\hat{B}(t), \partial_t\hat{B}(t)\right]\right] + \ldots. \end{aligned} \tag{50}$$

Calculating the first commutator in this series, we obtain

$$\left[\hat{B}(t), \partial_t \hat{B}(t)\right] = \frac{1}{c^2}\int_0^t \mathrm{d}t' \iint \left\{ j_i(x')\left[\hat{A}^i(x'), \hat{A}^j(x)\right] j_j(x)\right\} \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r}'. \tag{51}$$

The nonequal-time commutation relations for the operators $\hat{A}^i(x)$ are given by equation (11). Then (51) takes the form

$$\left[\hat{B}(t), \partial_t \hat{B}(t)\right] = -\frac{i}{c^2}\int_0^t \mathrm{d}t' \int \mathrm{d}\mathbf{r}\; j_j(x) \int \mathrm{d}\mathbf{r}' j_i(x')\, \delta_\perp^{ij} D_0(x - x'). \tag{52}$$

We suppose, as usual, that currents under consideration vanish at spatial infinities. In this case,

$$\int \mathrm{d}\mathbf{r}'\, j_i(x')\, \delta_\perp^{ij} D_0(x - x') = \int \mathrm{d}\mathbf{r}'\, D_0(x - x')\, \delta_\perp^{ij} j_i(x'). \tag{53}$$

Then, recalling the definition of $\tilde{A}^i(x)$ from the evolution operator equation (9), we obtain

$$\left[\hat{B}(t), \partial_t \hat{B}(t)\right] = -\frac{i}{c}\int j_i(x)\, \tilde{A}^i(x)\, \mathrm{d}\mathbf{r}. \tag{54}$$

Since the right-hand side of equation (54) is not an operator, the only first commutator in the right-hand side of equation (50) survives. Substituting equation (50) and equation (54) into equation (49) and then integrating over s, we find

$$\hat{K}(t) = \frac{1}{c}\int \left[j_i(x)\, \hat{A}^i(x) + \frac{1}{2}\, j_0(x)\, A^0(x)\right] \mathrm{d}\mathbf{r}. \tag{55}$$

Using the fact that in the Coulomb gauge

$$\exp\left[-i\hbar^{-1}\hat{H}_\gamma t\right] \hat{K}(t) = \frac{1}{c}\int \left[j_i(x)\hat{A}^i(\mathbf{r}) + \frac{1}{2}\, j_0(x)\, A^0(x)\right] \mathrm{d}\mathbf{r} \times \exp\left[-i\hbar^{-1}\hat{H}_\gamma t\right], \tag{56}$$

and taking into account equation (48) we make sure that state vector (8) does satisfy equation (7).

APPENDIX *B*
Total energy of photon radiation

Here we show that the sum (28) can be calculated analytically with the help of representation (26). We start at the definition of $W(N,t)$ from equation (26),

$$W(N,t) = \hbar c (N!)^{-1} \sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\ldots\sum_{\lambda_N=1}^{2}\int \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \ldots \mathrm{d}\mathbf{k}_N \times \left[\sum_{j=1}^{N} |\mathbf{k}_j|\right] \prod_{i=1}^{N} |y_{\mathbf{k}_i\lambda_i}|^2. \tag{57}$$

We first consider the term with $j = 1$. In the entire integrand (57), only the factor $|\mathbf{k}_1||y_{\mathbf{k}_1\lambda_1}|^2$ depends on λ_1 and $\mathbf{k}_1$. Therefore, everything except the factor $|\mathbf{k}_1||y_{\mathbf{k}_1\lambda_1}\hat{2}$ can be taken out from the signs of the sum over λ_1 and the integral over $\mathrm{d}\mathbf{k}_1$. Since the indices i are dummy (the limits of all summations and integrations are the same), we can cyclically shift their numbering ($i \to i-1$, *i.e.* $2 \to 1$, $3 \to 2, \ldots$, $N \to N-1$, $1 \to N$). We do the same with each term from the sum $j = 2, 3, 4, \ldots, N-1$. Now it is obvious that the sum over j in (57) degenerates into a factor N, and the quantity $W(N,t)$ takes the form

$$W(N,t) = \frac{\hbar c}{(N-1)!}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\ldots\sum_{\lambda_N=1}^{2}\int \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \ldots \mathrm{d}\mathbf{k}_N |\mathbf{k}_N| \times \prod_{i=1}^{N} |y_{\mathbf{k}_i\lambda_i}|^2. \tag{58}$$

It is easy to see that equation (58) can be written as

$$W(N,t) = \frac{\hbar c}{(N-1)!}\sum_{\lambda_N=1}^{2}\int \mathrm{d}\mathbf{k}_N |\mathbf{k}_N||y_{\mathbf{k}_N\lambda_N}|^2 \times \prod_{i=2}^{N}\left[\sum_{\lambda_i=1}^{2}\int \mathrm{d}\mathbf{k}_i |y_{\mathbf{k}_i\lambda_i}|^2\right]. \tag{59}$$

Finally, getting rid of dummy indices, we obtain

$$W(N,t) = \frac{A}{(N-1)!}\left[\sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\, |y_{\mathbf{k}\lambda}|^2\right]^{N-1},$$

$$A = \hbar c \sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\, k_0 |y_{\mathbf{k}\lambda}|^2, \qquad k_0 = |\mathbf{k}|. \tag{60}$$

The total energy $W(t)$ reads

$$W(t) = \sum_{N=1}^{\infty} W(N,t) = A\sum_{N=1}^{\infty}[(N-1)!]^{-1}\left[\sum_{\lambda=1}^{2}\int \mathrm{d}\mathbf{k}\, |y_{\mathbf{k}\lambda}|^2\right]^{N-1}. \tag{61}$$

The sum over N can be reduced to an exponent by the change $N = M - 1$. Thus, we justify equation (29).

APPENDIX *C*
Some details on one-photon radiation calculations

Functions $y_{\mathbf{k}\lambda}$ (13) for the current (31) have the form

$$y_{\mathbf{k}\lambda} = iq\int_0^t \mathrm{d}t' \frac{\mathbf{v}(t')\, \boldsymbol{\epsilon}^*_{\mathbf{k}\lambda}}{\left[\hbar c k_0 (2\pi)^2\right]^{1/2}} \exp\left\{i\left[k_0 c t' - \mathbf{k}\mathbf{r}(t')\right]\right\}, \tag{62}$$

$$\mathbf{k} = \left(k_\perp \cos\varphi, k_\perp \sin\varphi, k_\parallel\right), \quad k_\perp = k_0 \sin\theta, \quad k_\parallel = k_0\cos\theta. \tag{63}$$

Here φ is the angle between the x axis and the projection of the vector $\mathbf{k}$ onto the xy plane, and θ is the angle between the z axis and $\mathbf{k}$. Thus,

$$W(1,t) = \frac{\hbar c}{(2\pi)^2}\int \mathrm{d}\mathbf{k}\, k_0 \left|\int \mathrm{d}t'\, \mathbf{j}(x')\, \boldsymbol{\epsilon}^*_{\mathbf{k}\lambda} \exp\left[ik_0 c t' - \mathbf{k}\mathbf{r}(t')\right]\right|^2. \tag{64}$$

Then

$$\exp[-i\mathbf{k}\mathbf{r}(t')] = \exp\left[-ik_\perp R \sin\tau\right],$$

$$\exp(ik_0ct') = \exp\left[ick_0\omega^{-1}(\varphi - \pi/2)\right]\exp\left(ick_0\omega^{-1}\tau\right),$$

$$\mathbf{v}(\tau) = \omega R[\cos(\tau+\varphi), \sin(\tau+\varphi), 0],$$

$$\tau = \tau_{\rm i} + \omega t', \quad \tau_{\rm i} = \pi/2 - \varphi, \quad \int_0^t {\rm d}t' \rightarrow \int_{\tau_{\rm i}}^{\tau_{\rm i}+\omega t} \omega^{-1}\,{\rm d}\tau. \quad (65)$$

In the case under consideration, we chose linear polarization vectors $\boldsymbol{\epsilon}_{\mathbf{k}\lambda}$ as

$$\boldsymbol{\epsilon}_{\mathbf{k}1} = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta),$$
$$\boldsymbol{\epsilon}_{\mathbf{k}2} = (-\sin\varphi, \cos\varphi, 0),$$

$$\boldsymbol{\epsilon}_{\mathbf{k}1}\boldsymbol{\epsilon}_{\mathbf{k}1} = \boldsymbol{\epsilon}_{\mathbf{k}2}\boldsymbol{\epsilon}_{\mathbf{k}2} = 1, \quad \boldsymbol{\epsilon}_{\mathbf{k}1}\boldsymbol{\epsilon}_{\mathbf{k}2} = \boldsymbol{\epsilon}_{\mathbf{k}1}\mathbf{k} = \boldsymbol{\epsilon}_{\mathbf{k}2}\mathbf{k} = 0. \quad (66)$$

One can easily verify that the following relations hold,

$$\mathbf{v}(t')\,\boldsymbol{\epsilon}^*_{\mathbf{k}1} = \omega R\cos\theta\cos\tau, \quad \mathbf{v}(t')\,\boldsymbol{\epsilon}^*_{\mathbf{k}2} = \omega R\sin\tau. \quad (67)$$

Now it follows from equation (62) that

$$y_{\mathbf{k}1} = \frac{iqR\cos\theta}{\left[k_0(2\pi)^2\hbar c\right]^{1/2}}\,Y_{\mathbf{k}}(\varphi) \times \int_{\tau_{\rm i}}^{\tau_{\rm i}+\omega t} {\rm d}\tau\,\exp\left(ick_0\omega^{-1}\tau\right)\cos\tau\exp(-ik_\perp R\sin\tau),$$

$$y_{\mathbf{k}2} = \frac{iqR}{\left[k_0(2\pi)^2\hbar c\right]^{1/2}}\,Y_{\mathbf{k}}(\varphi) \times \int_{\tau_{\rm i}}^{\tau_{\rm i}+\omega t} {\rm d}\tau\,\exp\left(ick_0\omega^{-1}\tau\right)\sin\tau\exp(-ik_\perp R\sin\tau),$$

$$Y_{\mathbf{k}}(\varphi) = \exp\left[ick_0\omega^{-1}(\varphi - \pi/2)\right]. \quad (68)$$

At this stage, we utilize a well known plane wave expansion of the Bessel functions $J_n(x)$ (see, for example, Sokolov & Ternov, 1986),

$$\exp(-ik_\perp R\sin\tau) = \sum_{n=-\infty}^{+\infty} J_n(k_\perp R)\exp(-in\tau),$$

$$\sin\tau\exp(-ik_\perp R\sin\tau) = i\sum_{n=-\infty}^{+\infty} J_n'(k_\perp R)\exp(-in\tau), \quad (69)$$

$$\cos\tau\exp(-ik_\perp R\sin\tau) = \sum_{n=-\infty}^{+\infty} \frac{n}{k_\perp R}J_n(k_\perp R)\exp(-in\tau).$$

Using (69) in (68), we obtain

$$y_{\mathbf{k}1} = i\frac{qR\cos\theta}{\left[k_0(2\pi)^2\hbar c\right]^{1/2}}\,Y_{\mathbf{k}}(\varphi)\sum_{n=-\infty}^{+\infty}\frac{nJ_n(k_\perp R)}{k_\perp R}\,F^n_{\mathbf{k}}(\varphi, t),$$

$$y_{\mathbf{k}2} = -\frac{qR}{\left[k_0(2\pi)^2\hbar c\right]^{1/2}}\,Y_{\mathbf{k}}(\varphi)\sum_{n=-\infty}^{+\infty} J_n'(k_\perp R)\,F^n_{\mathbf{k}}(\varphi, t),$$

$$F^n_{\mathbf{k}}(\varphi, t) = \int_{\tau_{\rm i}}^{\tau_{\rm i}+\omega t} {\rm d}\tau\,\exp\left[i\left(ck_0\omega^{-1} - n\right)\tau\right], \quad (70)$$

we can rewrite equation (70) as follows,

$$y_{\mathbf{k}1} = \frac{iq\cot\theta}{\left[k_0^3(2\pi)^2\hbar c\right]^{1/2}}\,Y_{\mathbf{k}}(\varphi)\sum_{n=-\infty}^{+\infty} nJ_n(k_\perp R)\,F^n_{\mathbf{k}}(\varphi, t),$$

$$y_{\mathbf{k}2} = -\frac{qR}{\left[k_0(2\pi)^2\hbar c\right]^{1/2}}\,Y_{\mathbf{k}}(\varphi)\sum_{n=-\infty}^{+\infty} J_n'(k_\perp R)\,F^n_{\mathbf{k}}(\varphi, t). \quad (71)$$

Now, we can calculate the corresponding probabilities $|y_{\mathbf{k}\lambda}|^2$,

$$\left|y_{\mathbf{k}1}\right|^2 = \frac{q^2}{\hbar c}\frac{\cot^2\theta}{k_0^3(2\pi)^2}\left|\sum_{n=-\infty}^{+\infty} nJ_n(k_\perp R)\,F^n_{\mathbf{k}}(\varphi, t)\right|^2,$$

$$\left|y_{\mathbf{k}2}\right|^2 = \frac{q^2}{\hbar c}\frac{R^2}{k_0(2\pi)^2}\left|\sum_{n=-\infty}^{+\infty} J_n'(k_\perp R)\,F^n_{\mathbf{k}}(\varphi, t)\right|^2. \quad (72)$$

The radiated energy (30) has to be calculated in the following manner,

$$W(1,t) = W_1(1,t) + W_2(1,t),$$

$$W_1(1,t) = \hbar c\int {\rm d}\mathbf{k}\,k_0\left|y_{\mathbf{k}1}\right|^2 = \int {\rm d}\mathbf{k}\,\frac{q^2\cot^2\theta}{k_0^2(2\pi)^2}\left|\sum_{n=-\infty}^{+\infty} nJ_n(k_\perp R)F^n_{\mathbf{k}}(\varphi, t)\right|^2,$$

$$W_2(1,t) = \hbar c\int {\rm d}\mathbf{k}\,k_0\left|y_{\mathbf{k}2}\right|^2 = \int {\rm d}\mathbf{k}\,\frac{q^2R^2}{(2\pi)^2}\left|\sum_{n=-\infty}^{+\infty} J_n'(k_\perp R)F^n_{\mathbf{k}}(\varphi, t)\right|^2. \quad (73)$$

Note that the functions $F^n_{\mathbf{k}}(\varphi, t)$ can be represented as

$$F^n_{\mathbf{k}}(\varphi, t) = \omega\exp\left[-i\left(ck_0\omega^{-1} - n\right)\varphi\right]\exp\left[i\frac{\pi}{2}\left(ck_0\omega^{-1} - n\right)\right] \times \int_0^t {\rm d}t'\exp\left[i(ck_0 - n\omega)\,t'\right]. \quad (74)$$

Using the well known integral representation of Kronecker's delta function,

$$\oint {\rm d}\varphi\,\exp[i(n-n')\,\varphi] = 2\pi\delta_{nn'}, \quad (75)$$

we can transform the quantities $W_1(1,t)$ and $W_2(1,t)$ as follows,

$$W_1(1,t) = q^2\omega^2\sum_{n=-\infty}^{+\infty}\int_0^\infty\frac{{\rm d}k_0}{2\pi}\int_0^\pi \sin\theta\,{\rm d}\theta\,\cot^2\theta\,n^2J_n^2(k_\perp R) \times \left|\int_0^t {\rm d}t'\,\exp\left[i(ck_0 - n\omega)\,t'\right]\right|^2,$$

$$W_2(1,t) = q^2\omega^2R^2 \sum_{n=-\infty}^{+\infty} \int_0^\infty \frac{\mathrm{d}k_0}{2\pi} \int_0^\pi \sin\theta\,\mathrm{d}\theta\, k_0^2 J_n'^2(k_\perp R) \times \left| \int_0^t \mathrm{d}t' \, \exp\big[i(ck_0 - n\omega)\,t'\big] \right|^2. \quad (76)$$

Subsituting the functions $W_1(1,t)$ and $W_2(1,t)$ into equation (73), we obtain the final result equation (32)

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